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Baxter $\mathbb{Q}$-operator and separation of variables for the open $\mathrm{SL}(2,\mathbb{R})$ spin chain

Sergey É. Derkachov

Department of Mathematics, St.-Petersburg Technology Institute
St.-Petersburg, Russia
E-mail: derkach@th.u-psud.fr

Gregory P. Korchemsky

Laboratoire de Physique Théorique[*], *Université de Paris XI*
91405 Orsay Cédex, France
E-mail: korchems@th.u-psud.fr

Alexander N. Manashov[†]

Institut für Theoretische Physik II, Ruhr-Universität Bochum
D-44780 Bochum, Germany
E-mail: manashov@tp2.ruhr-uni-bochum.de

Abstract: We construct the Baxter $\mathbb{Q}$-operator and the representation of the Separated Variables (SoV) for the homogeneous open $\mathrm{SL}(2,\mathbb{R})$ spin chain. Applying the diagrammatical approach, we calculate Sklyanin's integration measure in the separated variables and obtain the solution to the spectral problem for the model in terms of the eigenvalues of the $\mathbb{Q}$-operator. We show that the transition kernel to the SoV representation is factorized into the product of certain operators each depending on a single separated variable. As a consequence, it has a universal pyramid-like form that has been already observed for various quantum integrable models such as periodic Toda chain, closed $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ spin chains.

Keywords: Integrable Hierarchies, Lattice Integrable Models, Bethe Ansatz.

[*]Unite Mixte de Recherche du CNRS (UMR 8627)

[†]Permanent address: Department of Theoretical Physics, Sankt-Petersburg State University, St.-Petersburg, Russia

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1. Introduction

Recently it has been found that the evolution equations describing the scale dependence of certain correlation functions in four-dimensional Yang-Mills theory possess a hidden symmetry. Remarkably enough, the emerging integrable structures are well-known in the theory of lattice integrable models [1] as corresponding to open Heisenberg spin magnets. In particular, the energy spectrum of the magnet determines the spectrum of the anomalous dimensions of the correlation functions in Yang-Mills theory [2, 3, 4]. A unusual feature of these models as compared with conventional magnets studied thoroughly in applications to statistical physics [5] is that the spin operators are generators of infinite-dimensional representations of the $\mathrm{SL}(2,\mathbb{R})$ group. This group emerges as subgroup of the full $\mathrm{SO}(4,2)$ conformal group of four-dimensional Yang-Mills theory.

Exact solution of the spectral problem for integrable systems with infinite-dimensional quantum space is a nontrivial task. The conventional Algebraic Bethe Ansatz (ABA) [6] is not always applicable to such systems and one has to rely instead on a more elaborated methods like the Baxter $\mathbb{Q}$-operator [5] and the Separation of Variables (SoV) [7]. Being combined together, the two methods allow one to find the energy spectrum of the model and obtain integral representation for the eigenstates. At present, such program has been carried out for a number of models with periodic boundary conditions. They include periodic Toda chain [8, 9], the DST model [10], noncompact closed SL(2) Heisenberg magnets [11, 12, 13] and Calogero-Sutherland model [14]. In the present paper, we apply the both methods to the quantum $\mathrm{SL}(2,\mathbb{R})$ open Heisenberg spin chain.

A systematic approach to building quantum integrable models with nontrivial boundary conditions (including open Heisenberg spin chains) has been developed by Sklyanin [15]. For such models a little progress has been made in constructing the $\mathbb{Q}$-operator and the SoV representation. One of the reasons for this is that the R-matrix formulation is more cumbersome in that case as compared to the models with periodic boundary conditions and, in addition, there exist no regular procedure for obtaining the $\mathbb{Q}$-operator.

In this paper, we construct the Baxter $\mathbb{Q}$-operator and representation of the Separated variables for the quantum $\mathrm{SL}(2,\mathbb{R})$ open spin chain. Our analysis is based on the Feynman diagram approach described at length in previous publications [12, 13]. In this approach, one realizes the $\mathbb{Q}$-operator as an integral operator acting on the quantum space of the model and represents its kernel as a certain Feynman diagram. Then, various properties of the $\mathbb{Q}$-operator can be established by making use of a few elementary diagrammatical relations. Using the obtained expressions, we determine the energy spectrum of the open Heisenberg spin chain in terms of the eigenvalues of the $\mathbb{Q}$-operator and obtain integral representation for the eigenfunctions.

The presentation is organized as follows. In section 2 we define the open Heisenberg magnet with the $\mathrm{SL}(2,\mathbb{R})$ spin symmetry and review Sklyanin's formulation of the model. In section 3 we construct Baxter $\mathbb{Q}$-operator for the homogeneous open spin chain and establish its properties. In section 4 we present an explicit construction of the unitary transformation to the Separated Variables for the open $\mathrm{SL}(2,\mathbb{R})$ spin chain. In particular, we calculate the integration measure defining the scalar product in the SoV representation and discuss its analytical properties. In section 5 we demonstrate that for the open spin chain with two sites the eigenvalues of the $\mathbb{Q}$-operator coincide with the Wilson orthogonal polynomials. Section 6 contains concluding remarks. Some technical details and description of the diagrammatical technique are given in appendix A.

2. Open Heisenberg spin chain

2.1 Definition of the model

The homogeneous open Heisenberg spin chain is a lattice model of N interacting spins $\vec{S}_n = (S_n^1, S_n^2, S_n^3)$ (with $n = 1,\ldots,N$) described by the hamiltonian

$$\mathcal{H}_N = \sum_{n=1}^{N-1} H_{n,n+1}\,, \qquad H_{n,n+1} = 2\left[\psi(J_{n,n+1}) - \psi(2s)\right], \tag{2.1}$$

where $\psi(x) = d\log\Gamma(z)/dz$ is the Euler ψ-function. The pairwise hamiltonian $H_{n,n+1}$ defines the interaction between two neighboring spins $\vec{S}_n$ and $\vec{S}_{n+1}$. It is expressed in terms of the operator $J_{n,n+1}$ related to their sum

$$J_{n,n+1}(J_{n,n+1}-1) = (\vec{S}_n + \vec{S}_{n+1})^2 \,. \tag{2.2}$$

The spin operators in different sites commute with each other and obey the standard commutation relations

$$[S_n^a, S_k^b] = i\varepsilon_{abc}\delta_{nk}S_n^c \,, \qquad \vec{S}_n^2 = s_n(s_n-1) \,. \tag{2.3}$$

We shall assume for simplicity that the spin chain is homogeneous, $s_1 = \cdots = s_N = s$, with real $s \geq 1/2$ the same as in (2.1).

Notice that the hamiltonian (2.1) does not involve interaction between the boundary spins $\vec{S}_1$ and $\vec{S}_N$. If one added the corresponding two-particle hamiltonian $H_{N,1}$ to the r.h.s. of (2.1), the resulting hamiltonian would define a homogeneous closed Heisenberg spin chain. The latter model admits solution within the R-matrix approach both by the ABA method [6] and the methods of the Baxter $\mathbb{Q}$-operator [11] and SoV [13]. In the present paper we extend the analysis performed in the papers [11, 13] to the case of the open spin chain and apply the method of the Baxter $\mathbb{Q}$-operator to solve the spectral problem for the hamiltonian (2.1)

$$\mathcal{H}_N \Psi_{\boldsymbol{q}}(z_1,\ldots,z_N) = E_{\boldsymbol{q}}\Psi_{\boldsymbol{q}}(z_1,\ldots,z_N) \,. \tag{2.4}$$

Here $\boldsymbol{q}$ denotes the complete set of the quantum numbers parameterizing the energy spectrum and z_n (with $n = 1,\ldots,N$) are the coordinates on the quantum space V_n associated with the nth site of the spin chain.

The hamiltonian $\mathcal{H}_N$ acts on the quantum space of the model $\mathcal{V}_N = \prod_{n=1}^N \otimes V_n$ and its energy spectrum depends on the choice of the Hilbert space V_n. In what follows we shall assume that $\mathcal{V}_N$ is spanned by functions $\Psi(z_1,\ldots,z_N) \in \mathcal{V}_N$ holomorphic in the upper half-plane $\operatorname{Im} z_n > 0$ and normalizable with respect to the scalar product

$$\langle\Psi_1|\Psi_2\rangle = \int \mathcal{D}^N z \; (\Psi_1(z_1,\ldots,z_N))^* \, \Psi_2(z_1,\ldots,z_N) \,, \tag{2.5}$$

where integration measure is defined as $\mathcal{D}^N z = \prod_{n=1}^N \mathcal{D}z_n$ with $(z_n = x_n + iy_n)$

$$\mathcal{D}z_n = \frac{2s-1}{\pi}\, d^2 z_n \, (2\operatorname{Im} z_n)^{2s-2}\theta(\operatorname{Im} z_n) = \frac{2s-1}{\pi}\, dx_n dy_n \, (2y_n)^{2s-2}\theta(y_n) \tag{2.6}$$

and integration in (2.5) goes over the upper half-plane. The spin operators $\vec{S}_n$ can be realized on this space as differential operators[1]

$$S_n^+ = z_n^2\partial_{z_n} + 2s\, z_n \,, \qquad S_n^- = -\partial_{z_n} \,, \qquad S_n^0 = z_n\partial_{z_n} + s \,. \tag{2.7}$$

[1]In Yang-Mills theory the spin operators (2.7) define representation of the generators of the collinear $\mathrm{SL}(2,\mathbb{R})$ subgroup of the full $\mathrm{SO}(4,2)$ conformal group on the space of correlation functions $\langle 0|\Phi_s(z_1 n)\cdots\Phi_s(z_N n)|0\rangle$ of primary fields with conformal spin s and "living" on the light-cone $n_\mu^2 = 0$.

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where $S_n^\pm = S_n^1 \pm i S_n^2$ and $S_n^0 = S_n^3$. These operators are anti-hermitian with respect to the scalar product (2.5)

$$(S_n^0)^\dagger = -S_n^0\,, \qquad (S_n^\pm)^\dagger = -S_n^\pm\,. \tag{2.8}$$

Notice that the quantum space of the model is infinite-dimensional for arbitrary finite N. For integer and half-integer s, the Hilbert space V_n coincides with the representation space of unitary representation of the $\mathrm{SL}(2,\mathbb{R})$ group of the discrete series [16].

2.2 R-matrix formulation

The open $\mathrm{SL}(2,\mathbb{R})$ Heisenberg spin magnet (2.1) is a completely integrable model. To identify its integrals of motion we follow Sklyanin's approach [15]. To begin with, one defines the Lax operator for the $\mathrm{SL}(2,\mathbb{R})$ magnet

$$L_n(u) = u + i(\vec{\sigma}\cdot\vec{S}_n) = \begin{pmatrix} u + iS_n^0 & iS_n^- \\ iS_n^+ & u - iS_n^0 \end{pmatrix}, \tag{2.9}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. It acts on the tensor product of the auxiliary space and the quantum space in the nth site, $\mathbb{C}^2 \otimes V_n$. Taking the product of N Lax operators along the spin chain in the auxiliary space one defines the operator — the monodromy matrix for the closed spin chain

$$T_N(u) = L_1(u)\cdots L_N(u) = \begin{pmatrix} a(u) & b(u) \\ c(u) & d(u) \end{pmatrix}, \tag{2.10}$$

which is a 2×2 matrix with the entries $a(u), \dots, d(u)$ being operators acting on $\mathcal{V}_N$. It satisfies the Yang-Baxter commutation relations

$$R_{12}(u-v)\overset{1}{T}_N(u)\overset{2}{T}_N(v) = \overset{2}{T}_N(v)\overset{1}{T}_N(u)R_{12}(u-v)\,, \tag{2.11}$$

where $\overset{1}{T}_N(u) = T_N(u)\otimes \mathbb{1}$ and $\overset{2}{T}_N(v) = \mathbb{1}\otimes T_N(v)$. The R-matrix acts on the tensor product of two auxiliary spaces, $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$R_{12}(u) = u\,\mathbb{1} + i\,P_{12}\,, \tag{2.12}$$

with P_{12} being the permutation operator. The monodromy matrix for the open spin chain is defined as [15][2]

$$\mathbb{T}_N(u) = T_N(u)\,T_N^{-1}(-u+i) = \frac{1}{\rho^N(u)}\cdot T_N(u)\,\sigma_2\,T_N^t(-u)\,\sigma_2\,, \tag{2.13}$$

where the c-valued factor $\rho(u) = (u-is)(u+i(s-1))$ absorbs all poles of $\mathbb{T}_N(u)$ and the superscript 't' denotes transposition in the auxiliary space. It satisfies the fundamental "reflection" Yang-Baxter relation [17, 15, 18]

$$\overset{2}{\mathbb{T}}_N(v)\,R_{12}(u+v-i)\,\overset{1}{\mathbb{T}}_N(u)\,R_{12}(u-v) = R_{12}(u-v)\,\overset{1}{\mathbb{T}}_N(u)\,R_{12}(u+v-i)\,\overset{2}{\mathbb{T}}_N(v) \tag{2.14}$$

[2] General definition of the integrable spin chain with nontrivial boundary conditions involves the boundary matrices $K_\pm$ [17, 15, 18]. The hamiltonian (2.1) corresponds to the simplest case $K_\pm = \mathbb{1}$.

with the same R-matrix (2.12). It proves convenient to change a normalization of $\mathbb{T}_N(u)$ as

$$\widehat{\mathbb{T}}_N(u) = \rho^N(-u)\,\mathbb{T}_N(-u) = T_N(-u)\,\sigma_2\,T_N^t(u)\,\sigma_2 = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{2.15}$$

It follows from (2.13) that $\widehat{\mathbb{T}}_N(u)$ satisfies the relation

$$\widehat{\mathbb{T}}_N(-u-i)\,\widehat{\mathbb{T}}_N(u) = [(u+is)(u-i(s-1))]^{2N}\,\mathbb{1}\,. \tag{2.16}$$

The Yang-Baxter relation (2.14) leads to the set of fundamental relations for the operators $A(u),\ldots,D(u)$. For our purposes we will need only two of them

$$\begin{aligned} B(u)B(v) &= B(v)B(u)\,, \\ B(u)D(v) &= \frac{(u+v+i)(u-v-i)}{(u-v)(u+v)}D(v)B(u) + i\left[A(u) + \frac{u+v+i}{u-v}D(u)\right]\frac{B(v)}{u+v}\,. \end{aligned} \tag{2.17}$$

The monodromy matrix (2.15) satisfies the following relation

$$\widehat{\mathbb{T}}_N(u) = \frac{1}{2u-i}\left[2u\,\sigma_2\widehat{\mathbb{T}}_N^t(-u)\sigma_2 - i\,\widehat{\mathbb{T}}_N(-u)\right]. \tag{2.18}$$

To verify it one starts with the definition of $\widehat{\mathbb{T}}_N(u)$, eq. (2.15), interchanges the operators $T_N(-u)$ and $T_N^t(u)$ with a help of the Yang-Baxter relation (2.11) and uses the explicit expression (2.12) for the R-matrix. Substitution of (2.15) into (2.18) yields

$$D(u) = \frac{1}{2u-i}\left[2uA(-u) - iD(-u)\right], \qquad \frac{B(u)}{2u+i} = \frac{B(-u)}{-2u+i}\,. \tag{2.19}$$

In the standard manner, the transfer matrix for the open spin chain $\widehat{t}_N(u)$ is defined as the trace of the monodromy matrix (2.15) over the auxiliary space

$$\widehat{t}_N(u) = \mathrm{tr}\,\widehat{\mathbb{T}}_N(u) = A(u) + D(u) = \left(1 - \frac{i}{2u}\right)D(u) + \left(1 + \frac{i}{2u}\right)D(-u)\,, \tag{2.20}$$

where in the last relation we took into account (2.19). Following Sklyanin [15] and making use of the Yang-Baxter relation (2.14), one can show that the transfer matrix commutes with itself for different values of the spectral parameter, with the hamiltonian (2.1) and with the operator of the total spin $\vec{S} = \sum_{n=1}^N \vec{S}_n$

$$[\widehat{t}_N(u), \widehat{t}_N(v)] = [\widehat{t}_N(u), \mathcal{H}_N] = [\widehat{t}_N(u), \vec{S}] = 0\,. \tag{2.21}$$

The expansion of $\widehat{t}_N(u)$ in powers of u generates the integrals of motion of the model. One deduces from (2.20) and (2.15) that the transfer matrix is an even polynomial in u of degree $2N$, $\widehat{t}_N(-u) = \widehat{t}_N(u)$, which scales at large u as $\widehat{t}_N(u) \sim 2(-1)^N u^{2N}$. In addition, it follows from (2.13) that $\mathbb{T}_N(i/2) = \mathbb{1}$ leading to

$$\widehat{t}_N\left(-\frac{i}{2}\right) = 2\rho^N\left(\frac{i}{2}\right) = 2\left(s - \frac{1}{2}\right)^{2N}. \tag{2.22}$$

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These properties imply that $t_N(u) - t_N(\pm i/2)$ is proportional to $(u+i/2)(u-i/2)$

$$\widehat{t}_N(u) = (-1)^N \left(u^2 + \frac{1}{4}\right) \left[2u^{2N-2} + \widehat{q}_2\, u^{2N-4} + \cdots + \widehat{q}_{N-1}\, u^2 + \widehat{q}_N\right] + 2\left(s - \frac{1}{2}\right)^{2N} . \tag{2.23}$$

Here the $\widehat{q}$-operators are given by polynomials in the spin operators $\vec{S}_n$, for instance

$$\widehat{q}_2 = -4\vec{S}^2 + 2Ns(s-1) - \frac{1}{2} . \tag{2.24}$$

It follows from (2.21) and (2.23) that $N-1$ operators $\widehat{q}_2, \dots, \widehat{q}_N$ form the family of mutually commuting SL(2) invariant integrals of motion. Since $[\mathcal{H}_N, \vec{S}] = 0$, the remaining Nth integral of motion is provided by one of the components of the total spin. It is convenient to choose the latter as $iS_- = -i\sum_n \partial_{z_n}$ since its eigenvalues define the total momentum.

Thus, the open Heisenberg spin chain is a completely integrable model and the spectral problem for the hamiltonian (2.1) can be reformulated as the spectral problem for the transfer matrix

$$\begin{aligned} \widehat{t}_N(u)\Psi_{\boldsymbol{q},p}(z_1,\dots,z_N) &= t_N(u)\Psi_{\boldsymbol{q},p}(z_1,\dots,z_N) , \\ (iS_- - p)\Psi_{\boldsymbol{q},p}(z_1,\dots,z_N) &= 0 , \end{aligned} \tag{2.25}$$

where $t_N(u)$ is the eigenvalue of the transfer matrix (2.20) and $\boldsymbol{q} = (q_2, \dots, q_N)$ denotes the eigenvalues of the integrals of motion. A general solution to (2.25) takes the form

$$\Psi_{\boldsymbol{q},p}(z_1,\dots,z_N) = \int_{-\infty}^{\infty} dx_0 \, \mathrm{e}^{ipx_0} \, \Psi_{\boldsymbol{q}}(z_1 - x_0, \dots, z_N - x_0) , \tag{2.26}$$

where integration goes along the real axis. The eigenstate $\Psi_{\boldsymbol{q}}(z_1,\dots,z_N)$ has to diagonalize simultaneously the operators $\widehat{q}_2, \dots, \widehat{q}_N$.

3. Baxter $\mathbb{Q}$-operator

To solve the spectral problem for the open Heisenberg spin chain, eq. (2.25), we apply the method of the Baxter $\mathbb{Q}$-operator. The method relies on the existence of the operator $\mathbb{Q}(u)$ which acts on the quantum space of the model $\mathcal{V}_N$, depends on the spectral parameter u and satisfies the following defining relations:

- Commutativity:

$$[\mathbb{Q}(u), \mathbb{Q}(v)] = 0 . \tag{3.1}$$

- *Q-t* relation:

$$\left[\mathbb{Q}(u), \widehat{t}_N(u)\right] = 0 . \tag{3.2}$$

- Baxter relation:

$$\widehat{t}_N(u)\, \mathbb{Q}(u) = \Delta_+(u)\, \mathbb{Q}(u+i) + \Delta_-(u)\, \mathbb{Q}(u-i) , \tag{3.3}$$

where $\Delta_\pm(u)$ are some scalar functions of u. For the homogeneous open spin chain they are given by

$$\Delta_\pm(u) = (-1)^N \frac{2u \mp i}{2u}(u \pm is)^{2N} \,. \tag{3.4}$$

In this section, we construct the operator $\mathbb{Q}(u)$ satisfying eqs. (3.1)–(3.3) and discuss its properties.

It follows from (3.1) and (3.2) that the Baxter $\mathbb{Q}$-operator and the transfer matrix $\widehat{t}_N(u)$ share the common set of the eigenstates

$$\mathbb{Q}(u)\Psi_{\boldsymbol{q}}(z_1,\ldots,z_N) = Q_{\boldsymbol{q}}(u)\Psi_{\boldsymbol{q}}(z_1,\ldots,z_N) \,. \tag{3.5}$$

The eigenstates $\Psi_{\boldsymbol{q}}(z_1,\ldots,z_N)$ are the solutions to the Schrödinger equation (2.4) whereas the corresponding eigenvalues of the $\mathbb{Q}$-operator, $Q_{\boldsymbol{q}}(u)$, satisfy the Baxter relation (3.3) with the transfer matrix $\widehat{t}_N(u)$, eq. (2.23), replaced by its eigenvalue. As we will show below, the Baxter $\mathbb{Q}$-operator encodes information about the spectrum of the open spin chain. Namely, having calculated its eigenvalues $Q_{\boldsymbol{q}}(u)$ one would be able to reconstruct the energy spectrum of the model $E_{\boldsymbol{q}}$.

3.1 Gauge transformations

Our approach to constructing the Baxter $\mathbb{Q}$-operator is based on the representation of $\mathbb{Q}(u)$ as an integral operator acting on the quantum space of the model

$$[\mathbb{Q}(u)\Psi]\,(z_1,\ldots,z_N) = \int \mathcal{D}^N w\, Q_u(z_1\ldots,z_N|\bar w_1,\ldots,\bar w_N)\Psi(w_1,\ldots,w_N)\,, \tag{3.6}$$

with $\bar w_n = w_n^*$ and the integration measure defined in (2.6). To find the explicit expression for the kernel $Q_u(z_1\ldots,z_N|\bar w_1,\ldots,\bar w_N)$ we shall explore the fact that the transfer matrix for the open spin chain, eq. (2.20), is invariant under local gauge transformations of the Lax operators [8, 11]

$$L_n(u) \to \widetilde{L}_n(u) = M_n^{-1}\,L_n(u)\,M_{n+1}\,, \tag{3.7}$$

where M_n (with $n = 1,\ldots,N+1$) are arbitrary 2×2 matrices with $\det M_n \neq 0$. According to eqs. (2.10), (2.13) and (2.15) the operators $T_N(u)$ and $\widehat{\mathbb{T}}_N(u)$ are transformed under (3.7) as

$$\begin{aligned} T_N(u) &\to \widetilde{T}_N(u) = M_1^{-1}\,T_N(u)\,M_{N+1}\,, \\ \widehat{\mathbb{T}}_N(u) &\to \widetilde{\widehat{\mathbb{T}}}_N(u) = M_1^{-1}\,\widehat{\mathbb{T}}_N(u)\,M_1\,, \end{aligned} \tag{3.8}$$

so that the transfer matrix $\widehat{t}_N(u) = \operatorname{tr}\widehat{\mathbb{T}}_N(u)$ stays invariant.

The gauge rotation of the Lax operator, eq. (3.7), has been used by Pasquier and Gaudin to construct the Baxter operator for the periodic Toda chain [8] and it was later applied to the closed spin chain in refs. [11, 13]. In the latter case, the transfer matrix equals $\operatorname{tr} T_N(u)$ and in order to preserve its invariance under the transformation (3.8) one had to impose periodic boundary conditions $M_{N+1} = M_1$. For the open spin chain the matrices M_{N+1} and M_1 can be arbitrary, $M_{N+1} \neq M_1$.[3] In spite of this difference, many results obtained in refs. [11, 13] for the closed spin chain are applicable to the open chain.

To begin with, let us introduce the function [11, 13]

$$Y_u(z_1,\dots,z_N|\bar{w}_1,\dots,\bar{w}_{N+1}) = \prod_{k=1}^{N} (z_k - \bar{w}_k)^{-s-iu} (z_k - \bar{w}_{k+1})^{-s+iu}\,, \tag{3.9}$$

which is a (anti)holomorphic function of the complex variables $\vec{z} = (z_1,\dots,z_N)$ and $\vec{w} = (\bar{w}_1,\dots,\bar{w}_{N+1})$ in the upper half-plane $\mathrm{Im}\, z_k > 0$ and $\mathrm{Im}\, w_n > 0$ (with $\bar{w}_n = w_n^*$). It satisfies the following relations

$$\begin{aligned} \widetilde{b}(u;\bar{w}_1,\bar{w}_{N+1})\, Y_u(\vec{z}|\vec{w}) &= 0\,,\\ \widetilde{a}(u;\bar{w}_1,\bar{w}_{N+1})\, Y_u(\vec{z}|\vec{w}) &= (u+is)^N\, Y_{u+i}(\vec{z}|\vec{w})\,,\\ \widetilde{d}(u;\bar{w}_1,\bar{w}_{N+1})\, Y_u(\vec{z}|\vec{w}) &= (u-is)^N\, Y_{u-i}(\vec{z}|\vec{w})\,, \end{aligned} \tag{3.10}$$

where the operators $\widetilde{a}(u),\dots,\widetilde{d}(u)$ are defined similarly to (2.10) as the entries of the gauge rotated transfer matrix $\widetilde{T}_N(u)$, eq. (3.8), with the M-matrices given by

$$M_1 = \begin{pmatrix} 1 & 1/\bar{w}_1 \\ 0 & 1 \end{pmatrix}, \qquad M_{N+1} = \begin{pmatrix} 1 & 1/\bar{w}_{N+1} \\ 0 & 1 \end{pmatrix}, \tag{3.11}$$

with $\bar{w}_{N+1} \neq \bar{w}_1$. They are given by linear combinations of the operators $a(u),\dots,d(u)$, eq. (2.10), with the coefficients depending on the gauge parameters $\bar{w}_{N+1}$ and $\bar{w}_1$, which are identified as the right arguments of the kernel $Y_u(z_1,\dots,z_N|\bar{w}_1,\dots,\bar{w}_{N+1})$. Similar relations hold between the entries of the monodromy matrices $\widehat{\mathbb{T}}_N(u)$ and $\widetilde{\widetilde{\mathbb{T}}}_N(u)$, eqs. (3.8) and (2.15), so that

$$\widetilde{A}(u;\bar{w}_1) + \widetilde{D}(u;\bar{w}_1) = A(u) + D(u)\,, \qquad \widetilde{B}(u;\bar{w}_1) = B(u) + \mathcal{O}\left(\frac{1}{\bar{w}_1}\right), \tag{3.12}$$

where we indicated explicitly the dependence on the gauge parameter $\bar{w}_1$. Eqs. (3.10) play a crucial rôle in our subsequent analysis. Their derivation can be found in [11, 13].

To proceed further let us express the entries of the monodromy matrix of the open chain, $\widetilde{B}(u)$ and $\widetilde{D}(u)$, in terms of those for the closed spin chain, $\widetilde{a}(u),\dots,\widetilde{d}(u)$. One finds from (3.8), (2.10) and (2.15)

$$\widetilde{B}(u) = \widetilde{b}(-u)\widetilde{a}(u) - \widetilde{a}(-u)\widetilde{b}(u)\,, \qquad \widetilde{D}(u) = \widetilde{d}(-u)\widetilde{a}(u) - \widetilde{c}(-u)\widetilde{b}(u)\,. \tag{3.13}$$

3.2 Kernel of the $\mathbb{Q}$-operator

We now turn to constructing the kernel of the Baxter $\mathbb{Q}$-operator and consider the following auxiliary operator $\mathbb{G}(u,v) : \mathcal{V}_N \longmapsto \mathcal{V}_N$ with the kernel given by the convolution of two Y-functions introduced in the subsection 3.1

$$\begin{aligned} G_{u,v}(z_1,\dots,z_N|\bar{w}_1,\dots,\bar{w}_N) = \mathrm{e}^{i\pi s(2N-1)} \int \mathcal{D}y_2 \cdots \int \mathcal{D}y_N Y_u(z_1,\dots,z_N|\bar{w}_1,\bar{y}_2,\dots,\bar{y}_N,\bar{w}_N) \times \\ \times Y_v(y_2,\dots,y_N|\bar{w}_1,\bar{w}_2,\dots,\bar{w}_N)\,. \end{aligned} \tag{3.14}$$

[3] Notice that the monodromy matrices $\widetilde{T}_N(u)$ and $\widetilde{\widetilde{\mathbb{T}}}_N(u)$ satisfy the Yang-Baxter relations, Eq (2.11) and (2.14), respectively. This follows immediately from the invariance of the R-matrix (2.12) under transformations $R \to U\,R\,U^{-1}$ with $U = M \otimes M$.

Here the integration measure $\mathcal{D}y_n$ is defined in (2.6) and the prefactor is introduced for the later convenience. Notice that two Y-functions in (3.14) have a different number of arguments and depend on the same variables $\bar{w}_1$ and $\bar{w}_N$.

Let us demonstrate that for $v = -u$ the operator $\mathbb{G}(u, v)$ satisfies the relations (3.1)–(3.3) and, therefore, it can be identified as the Baxter $\mathbb{Q}$-operator for the homogeneous open Heisenberg spin chain

$$\mathbb{Q}(u) = \mathbb{G}(u, -u)\,, \tag{3.15}$$

or equivalently

$$\begin{aligned} Q_u(z_1, \ldots, z_N|\bar{w}_1, \ldots, \bar{w}_N) = {} & \mathrm{e}^{i\pi s(2N-1)}\,(z_1 - \bar{w}_1)^{-\beta_u}(z_N - \bar{w}_N)^{-\alpha_u} \times \\ & \times \prod_{n=2}^{N} \int \mathcal{D}y_n (z_{n-1} - \bar{y}_n)^{-\alpha_u}(z_n - \bar{y}_n)^{-\beta_u} \times \\ & \times (y_n - \bar{w}_{n-1})^{-\alpha_u}(y_n - \bar{w}_n)^{-\beta_u} \end{aligned} \tag{3.16}$$

with $\alpha_u = s - iu$ and $\beta_u = s + iu$. To prove (3.15) we apply the diagrammatical approach developed in ref. [13]. In this approach, one represents the kernel $G_{u,v}(\vec{z}|\vec{w})$, eq. (3.14), as the Feynman diagram shown in figure 1. There, the arrow with the index α that goes from y to z represents the factor $(z - \bar{y})^{-\alpha}$ (see eq. (A.1)) while the black blob denotes integration over the position w of the corresponding vertex with the $\mathrm{SL}(2,\mathbb{R})$ measure $\mathcal{D}w$ (see eq. (A.2) and figure 9).

The operator $\mathbb{G}(u, v)$ is symmetric under interchange of the spectral parameters

$$\mathbb{G}(u, v) = \mathbb{G}(v, u)\,, \tag{3.17}$$

or equivalently $G_{u,v}(\vec{z}|\vec{w}) = G_{v,u}(\vec{z}|\vec{w})$. The proof of (3.17) is based on the permutation identity shown in figure 4. Writing $\beta_u = \beta_v + i(u - v)$, one replaces the left-most vertical line in the left diagram in figure 1 by two lines with the indices β_v and $i(u - v)$. Then, one displaces the line with the index $i(u - v)$ across the diagram to the right with a help of the permutation identity until it merges with the right-most vertical line and changes its index to $\alpha_u + i(u - v) = \alpha_v$ (see ref. [13] for details). The resulting diagram coincides with the original one but with the spectral parameters interchanged. Furthermore, it follows from (3.14) and (3.10) that

$$\begin{aligned} \widetilde{a}(u; \bar{w}_1, \bar{w}_N)\, G_{u,v}(\vec{z}; \vec{w}) &= (u + is)^N G_{u+i,v}(\vec{z}; \vec{w})\,, \\ \widetilde{d}(u; \bar{w}_1, \bar{w}_N)\, G_{u,v}(\vec{z}; \vec{w}) &= (u - is)^N G_{u-i,v}(\vec{z}; \vec{w})\,, \\ \widetilde{b}(u; \bar{w}_1, \bar{w}_N)\, G_{u,v}(\vec{z}; \vec{w}) &= 0\,, \end{aligned} \tag{3.18}$$

where $\vec{z} = (z_1, \ldots, z_N)$ and $\vec{w} = (\bar{w}_1, \ldots, \bar{w}_N)$. Here the operators $\widetilde{a}(u), \ldots, \widetilde{d}(u)$ depend on the gauge parameters $\bar{w}_1$ and $\bar{w}_N$, which coincide with the corresponding arguments of the kernel $G_{u,v}(z_1, \ldots, z_N|\bar{w}_1, \ldots, \bar{w}_N)$.

Let us demonstrate that the operator $\mathbb{Q}(u)$ satisfies the Baxter equation (3.3). To this end, one examines the expression entering the l.h.s. of the Baxter equation (3.3) and applies eqs. (2.20), (3.12) and (3.15) to get

$$\widehat{t}_N(u)\,\mathbb{Q}(u) = (\widetilde{A}(u) + \widetilde{D}(u))\,\mathbb{G}(u, -u) = \left[\frac{2u - i}{2u}\,\widetilde{D}(u) + \frac{2u + i}{2u}\,\widetilde{D}(-u)\right]\mathbb{G}(u, -u)\,. \tag{3.19}$$

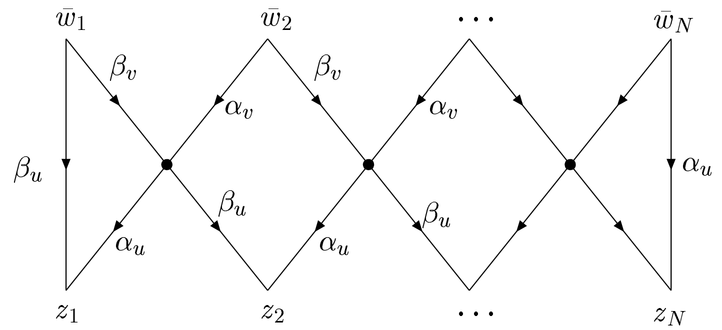

Figure 1: Diagrammatical representation of the function $G_{u,v}(\vec{z}|\vec{w})$. For $v = -u$ the diagram defines the kernel of the Baxter $\mathbb{Q}$-operator, eq. (3.16). Here $\alpha_x = s - ix$ and $\beta_x = s + ix$.

Taking into account eqs. (3.13), (3.18) and (3.17), one finds

$$\begin{aligned}\widetilde{D}(u)\,\mathbb{G}(u,-u) &= \widetilde{d}(-u)\widetilde{a}(u)\mathbb{G}(u,-u) = (u+is)^N\widetilde{d}(-u)\mathbb{G}(-u,u+i)\\ &= (u+is)^N(-u-is)^N\mathbb{G}(-u-i,u+i)\,.\end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.19) one concludes that the operator $\mathbb{Q}(u)$ defined in (3.16) verifies the Baxter relation (3.3). In addition, one deduces from (3.13) and (3.18) that the kernel of the $\mathbb{Q}$-operator is nullified by the operator $\widetilde{B}(u)$

$$\widetilde{B}(u;w_1)\,Q_u(z_1,\ldots,z_N|\bar{w}_1,\ldots,\bar{w}_N) = 0\,. \tag{3.21}$$

We will use this property in section 4 to construct the unitary transformation to the SoV representation.

The operator $\mathbb{Q}(u)$ has the following properties

- Parity:
$$\mathbb{Q}(u) = \mathbb{Q}(-u)\,. \tag{3.22}$$

- Normalization:
$$\mathbb{Q}(\pm is) = \mathbb{K}\,. \tag{3.23}$$

- Hermiticity:
$$(\mathbb{Q}(u))^\dagger = \mathbb{Q}(u^*)\,. \tag{3.24}$$

- SL(2) invariance:
$$[\mathbb{Q}(u),\vec{S}] = 0\,, \tag{3.25}$$

where $\mathbb{K}$ is the unit operator on the Hilbert space of the model $\mathcal{V}_N$ (see eq. (A.4) in appendix A) and $\vec{S} = \sum_{k=1}^N \vec{S}_k$ is the operator of the total spin. Eq. (3.22) is a consequence of (3.15) and (3.17). Eq. (3.23) follows from the fact that $\beta_{is} = \alpha_{-is} = 0$ so that the corresponding lines in the diagram in figure 1 disappear leading to drastic simplification of the kernel. Eq. (3.24) follows directly from the definition of the conjugated operator

$(\mathbb{Q}(u))^\dagger$. To verify (3.25) one notices that the kernel of the $\mathbb{Q}$-operator, eq. (3.16), is transformed under the $\mathrm{SL}(2,\mathbb{R})$ transformations as

$$Q_u(\vec{z}'|\vec{w}') = \prod_{k=1}^{N} (\gamma\bar{w}_k + \delta)^{2s} (\gamma z_k + \delta)^{2s}\, Q_u(\vec{z}|\vec{w}) \tag{3.26}$$

where $z'_k = (\alpha z_k + \beta)/(\gamma z_k + \delta)$ and $\bar{w}'_k = (\alpha\bar{w}_k + \beta)/(\gamma\bar{w}_k + \delta)$ with real $\alpha, \dots, \delta$ such that $\alpha\delta - \beta\gamma = 1$.

We are now ready to demonstrate that the $\mathbb{Q}$-operator (3.15) satisfies the relations (3.2) and (3.1). To verify (3.2), one performs the hermitean conjugation of the both sides of the Baxter equation (3.3). Taking into account (3.24), one finds that the r.h.s. of (3.3) goes into $\widehat{t}_N(u^*)\mathbb{Q}(u^*)$ whereas its l.h.s. is replaced by $\left(\widehat{t}_N(u)\mathbb{Q}(u)\right)^\dagger = \mathbb{Q}(u^*)\widehat{t}_N(u^*)$. Equating the two expressions one arrives at (3.2). Finally, let us show that the operator (3.15) satisfies the commutativity condition (3.1). The proof can be performed diagrammatically. To this end, one examines the Feynman diagram corresponding to the product $\mathbb{Q}(v)\mathbb{Q}(u) = \mathbb{G}(v,-v)\,\mathbb{G}(u,-u)$ and inserts a pair of lines with the indices $\pm i(u+v)$ into one of the central rhombuses as shown in figure 2. Displacing the two lines horizontally in the opposite directions with a help of the permutation identity (see figure 4) one obtains the Feynman diagrams shown in figure 2 to the right. It differs from the original diagram in that various α- and β-indices got interchanged and two additional lines with the indices $\pm i(u+v)$ connect the "end points", $\bar{w}_1$ with z_1 and $\bar{w}_N$ with z_N. Taking into account the definition of the $\mathbb{G}$-operator, eq. (3.14) (see figure 1) one finds that the Feynman integral corresponding to this diagram can be written as

$$[\mathbb{Q}(v)\,\mathbb{Q}(u)](\vec{z};\vec{w}) = (z_1 - \bar{w}_1)^{i(u+v)}(z_N - \bar{w}_N)^{-i(u+v)} \left[(\mathbb{G}(u^*,v^*))^\dagger\, \mathbb{G}(-u,-v)\right](\vec{z};\vec{w})\,, \tag{3.27}$$

where the kernel of the integral operator $(\mathbb{G}(u^*,v^*))^\dagger$ is given by $(G_{u^*,v^*}(w_1,\dots,w_N|\bar{z}_1, \dots, \bar{z}_N))^*$. According to (3.17), the r.h.s. of (3.27) is invariant under interchanging $u \leftrightarrows v$ thus proving the commutativity relation (3.1).

3.3 Contour-integral representation for the $\mathbb{Q}$-operator

In the previous subsection we constructed the Baxter $\mathbb{Q}$-operator for the homogeneous open spin chain, eqs. (3.16). As was already mentioned, the $\mathbb{Q}$-operator is diagonalized by the eigenstates of the model, eq. (3.5), and the corresponding eigenvalues $Q_{\boldsymbol{q}}(u)$ satisfy (3.3).

The Baxter equation (3.3) is a finite-difference functional equation and its solutions are defined up to multiplication by an arbitrary periodic function, $f(u+i) = f(u)$. To fix this ambiguity and determine eigenvalues of the $\mathbb{Q}$-operator, one has to specify analytical properties of $Q_{\boldsymbol{q}}(u)$. They can be identified using the following contour-integral representation for the $\mathbb{Q}$-operator on the quantum space of the model $\Psi(z_1,\dots,z_N) \in \mathcal{V}_N$

$$\begin{aligned}\left[\mathbb{Q}(u)\Psi\right](z_1,\dots,z_N) &= [B(s+iu, s-iu)]^{-2N+1} \int_0^1 \prod_{n=1}^{N} d\sigma_n\, (1-\sigma_n)^{s+iu-1}\, \sigma_n^{s-iu-1} \times \\ &\times \int_0^1 \prod_{k=2}^{N} d\tau_k\, (1-\tau_k)^{s+iu-1}\, \tau_k^{s-iu-1}\, \Psi(Z_1,\dots,Z_N)\,,\end{aligned} \tag{3.28}$$

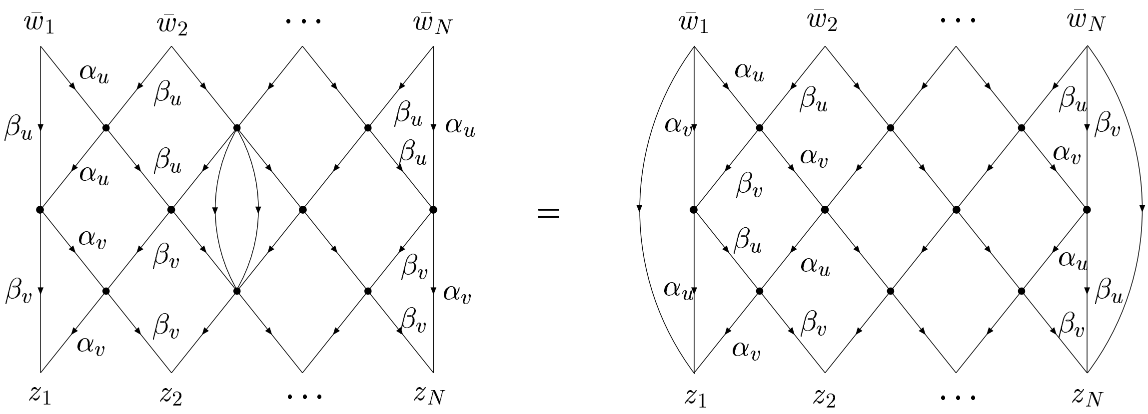

Figure 2: Diagrammatical proof of eq. (3.27). The left diagram represents the kernel of the operator $\mathbb{Q}(v)\mathbb{Q}(u)$. The right diagram is obtained by displacing two wavy lines carrying the indices $\pm i(u+v)$ to the right/left with a help of the permutation identity. Here $\alpha_x = s - ix$, $\beta_x = \alpha_{-x} = s + ix$.

where $B(x,y)$ is the Euler beta-function and the Z-coordinates are defined as

$$\begin{aligned} Z_1 &= (1-\sigma_1)z_1 + \sigma_1[\tau_2 z_1 + (1-\tau_2)z_2] \\ Z_k &= (1-\sigma_k)[\tau_k z_{k-1} + (1-\tau_k)z_k] + \sigma_k[\tau_{k+1}z_k + (1-\tau_{k+1})z_{k+1}]\,, \qquad (1<k<N)\,, \\ Z_N &= (1-\sigma_N)[\tau_N z_{N-1} + (1-\tau_N)z_N] + \sigma_N z_N\,. \end{aligned} \tag{3.29}$$

To obtain (3.28), one uses the integral representation for the $\mathbb{Q}$-operator, eq. (3.16), and applies the identity (A.5).

Since the function $\Psi(Z_1,\ldots,Z_N)$ is holomorphic in the upper half-plane $\operatorname{Im} z_n > 0$, the integral in the r.h.s. of (3.28) is convergent inside the strip $-s < \operatorname{Re}(iu) < s$ in the complex u-plane. Analytically continuing the integral outside this strip, one finds that it contains poles of the order $p \le 2N-1$ originating from integration at the vicinity of the end-points $\sigma_n\,,\tau_k \to 0,1$. They are located at $iu = \pm(n-s)$ (with n nonnegative integer) and are compensated by the beta-function prefactor entering (3.28). As a result, $[\mathbb{Q}(u)\Psi](z_1,\ldots,z_N)$ does not have poles in u and, therefore, the eigenvalues of the Baxter operator, $Q_{\boldsymbol{q}}(u)$, are entire functions of u.

Eq. (3.28) also allows one to determine asymptotic behaviour of $Q_{\boldsymbol{q}}(u)$ at large u. It is given by

$$Q_{\boldsymbol{q}}(u) \sim u^{2h}\left[1 + \mathcal{O}\left(\frac{1}{u^2}\right)\right], \tag{3.30}$$

with h nonnegative integer defining the total spin of the model

$$[\vec{S}^2 - (h+Ns)(h+Ns-1)]\Psi_{\boldsymbol{q}}(\vec{z}) = 0\,. \tag{3.31}$$

To establish (3.30), one verifies using (3.28) that the Baxter operator is invariant under arbitrary $\mathrm{SL}(2,\mathbb{C})$ transformations, in particular under the following one $z \mapsto -i(w+i)/(w-i)$

$$\Psi(z_1,\ldots,z_N) \mapsto \widetilde{\Psi}(w_1,\ldots,w_N) = \prod_{k=1}^{N}(w_k - i)^{-2s}\,\Psi\left(-i\frac{w_1+i}{w_1-i},\ldots,-i\frac{w_N+i}{w_N-i}\right), \tag{3.32}$$

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which map the upper half-plane $\mathrm{Im}\, z_k > 0$ into a unit disk $|w_k| < 1$. The main advantage of dealing with functions $\widetilde{\Psi}(w_1, \dots, w_N)$ holomorphic inside the unit circle is that solutions to (3.31) have a simple form in that case. Namely, the Hilbert space of the model contains the highest weights which satisfy (3.31) and are given by homogeneous translation invariant polynomials of degree h, $\widetilde{\Psi}(w_1, \dots, w_N) = P_h(w_1 - w_2, \dots, w_{N-1} - w_N)$. Since the Baxter operator (3.28) remains invariant under (3.32), one can substitute the function $\Psi(\vec{z})$ in (3.28) by such polynomial. Then, $\Psi(Z_1, \dots, Z_N)$ entering (3.28) becomes a polynomial in the σ- and τ-parameters. Integrating term-by-term in the r.h.s. (3.28) one finds that the dominant contribution at large u comes from terms containing a maximum number of σ's and τ's. This number equals $2h$ and leads to the asymptotics (3.30).

Given that $Q_{\boldsymbol{q}}(u)$ is an even function of u, eq. (3.22), and making use of (3.24), we conclude that the eigenvalues of the $\mathbb{Q}$-operator are real polynomials in u^2 of degree h

$$Q_{\boldsymbol{q}}(u) = a_{\boldsymbol{q}} \prod_{k=1}^{h} (u^2 - \lambda_k^2)\,, \qquad (Q_{\boldsymbol{q}}(u))^* = Q_{\boldsymbol{q}}(u^*) \tag{3.33}$$

with the normalization constant $a_{\boldsymbol{q}}$ fixed by the condition $Q_{\boldsymbol{q}}(is) = 1$, eq. (3.23). Substituting (3.33) into (3.3) and putting $u = \lambda_k^2$, one finds that the roots λ_k^2 satisfy the Bethe equations for the open spin chain [15].

3.4 Relation to the hamiltonian

Let us demonstrate that the hamiltonian of the open spin chain, eq. (2.1), is given by a logarithmic derivative of the Baxter operator evaluated at special values of the spectral parameter $u = \pm is$. Due to (3.23) the expansion of the $\mathbb{Q}$-operator around $u = \pm is$ can be written as

$$[\mathbb{Q}(\pm is + \epsilon)\Psi]\,(\vec{z}) = \Psi(\vec{z}) \mp i\epsilon\, [\mathcal{H}_N\, \Psi]\,(\vec{z}) + \mathcal{O}(\epsilon^2)\,, \tag{3.34}$$

with $\mathcal{H}_N$ being some integral operator. Its explicit form can be found from the contour-integral representation for the $\mathbb{Q}$-operator, (3.28). At $u = -is + \epsilon$ the beta-prefactor in the r.h.s. of (3.28) vanishes as ϵ^{2N-1} but it is compensated by poles coming from integration at the vicinity of $\sigma_n = \tau_k = 0$. Carefully separating contribution from this region, one obtains that the operator $\mathcal{H}_N$ entering (3.34) is given by the sum of two-particle integral operators

$$\mathcal{H}_N = -i\frac{d}{d\epsilon} \ln \mathbb{Q}(-is + \epsilon)\bigg|_{\epsilon=0} = \sum_{n=0}^{N-1} H_{n,n+1}\,, \tag{3.35}$$

where $H_{n,n+1}$ acts on $\Psi(z_n, z_{n+1}) \in V_n \otimes V_{n+1}$ as

$$[H_{12}\,\Psi](z_1, z_2) = -\int_0^1 \frac{d\tau}{\tau}(1-\tau)^{2s-1}\,[\Psi(z_{12}(\tau), z_2) + \Psi(z_1, z_{21}(\tau)) - 2\Psi(z_1, z_2)]\,, \tag{3.36}$$

with $z_{ik}(\tau) = (1-\tau)z_i + \tau z_k$. It is straightforward to check that the hamiltonian $H_{n,n+1}$ commutes with the two-particle spin $\vec{S}_n + \vec{S}_{n+1}$ defined in (2.7) and, therefore, it only depends on the Casimir operator $J_{n,n+1}$, eq. (2.2). To find the explicit form of this dependence one applies H_{12} to the state $\Psi(z_1, z_2) = (z_1 - z_2)^h/((z_1 + i)(z_2 + i))^{h+2s}$ with h nonnegative integer.[4] It diagonalizes simultaneously the Casimir operator $J_{12}\,\Psi = (h+2s)\Psi$ and the two-particle kernel $H_{12}\Psi = 2[\psi(h+2s) - \psi(2s)]\Psi$ leading to the expression

$$H_{n,n+1} = 2\left[\psi(J_{n,n+1}) \;-\; \psi(2s)\right], \tag{3.37}$$

which coincides with (2.1).

Eq. (3.35) establishes the relation between the hamiltonian of the model (2.1) and the Baxter $\mathbb{Q}$-operator. Obviously, the same relation holds between their eigenvalues

$$E_{\boldsymbol{q}} = \pm i \frac{d}{d\epsilon} \ln Q_{\boldsymbol{q}}(\pm is + \epsilon)\bigg|_{\epsilon=0}. \tag{3.38}$$

Thus, to reconstruct the energy spectrum of the model, one has to find polynomial solutions to the Baxter equation (3.3) and apply (3.38).

4. Separation of variables

In this section we will construct integral representation for the eigenstates of the model, eq. (2.4), by going over to the representation of the Separated Variables $(p,\boldsymbol{x}) = (p, x_1, \dots, x_{N-1})$ (SoV)

$$\Psi_{\boldsymbol{q},p}\,(z_1,\dots,z_N) = \int_{\mathbb{R}_+^{N-1}} d^{N-1}\boldsymbol{x}\,\mu(\boldsymbol{x})\,U_{p,\boldsymbol{x}}(z_1,\dots,z_N)\,\Phi_{\boldsymbol{q}}(\boldsymbol{x})\,. \tag{4.1}$$

Here $\Phi_{\boldsymbol{q}}(\boldsymbol{x})$ is the eigenfunction of the model in the separated variables. It is factorized into a product of functions depending on a single variable $\Phi_{\boldsymbol{q}}(\boldsymbol{x}) \sim Q(x_1)\dots Q(x_{N-1})$. As will be shown in this section, $Q(x_k)$ coincides with the eigenvalue of the Baxter $\mathbb{Q}$-operator. The kernel $U_{p,\boldsymbol{x}}$ of the unitary operator corresponding to the SoV transformation is defined as

$$U_{p,\boldsymbol{x}}(z_1,\dots,z_N) = \langle z_1,\dots,z_N|p,\boldsymbol{x}\rangle\,. \tag{4.2}$$

We will argue below that the separated variables x_k (with $k = 1,\dots,N-1$) take real positive values so that integration in (4.1) goes over $\boldsymbol{x}\in\mathbb{R}_+^{N-1}$ with $d^{N-1}\boldsymbol{x} = dx_1\cdots dx_{N-1}$ and $\mu(\boldsymbol{x})$ being a nontrivial integration measure. Eq. (4.1) defines the transformation $\Phi_{\boldsymbol{q}} \mapsto \Psi_{\boldsymbol{q},p}$. The inverse transformation looks as follows

$$\Phi_{\boldsymbol{q}}(\boldsymbol{x})\,\delta(p-p') = \langle p',\boldsymbol{x}|\Psi_{\boldsymbol{q},p}\rangle = \int \mathcal{D}^N z\,\left(U_{p',\boldsymbol{x}}(z_1,\dots,z_N)\right)^*\Psi_{\boldsymbol{q},p}\,(z_1,\dots,z_N)\,. \tag{4.3}$$

To construct the unitary transformation to the SoV representation one has to specify the complete set of the states $|p,\boldsymbol{x}\rangle$ and define the corresponding kernel (4.2). It is well-known that for the SL(2) spin chain with periodic boundary conditions, within the framework of the Sklyanin's approach [7], the basis vectors $|p,\boldsymbol{x}\rangle$ can be defined as eigenvectors of the operator $b(u)$ which is the off-diagonal matrix element of the monodromy matrix $T_N(u)$, eq. (2.10). We will demonstrate below that the same recipe also works for the open spin chain. Namely, the basis vectors $|p,\boldsymbol{x}\rangle$ in (4.2) can be defined as the eigenstates of the operator $B(u)$ entering the expression for monodromy matrix $\mathbb{T}_N(u)$, eq. (2.15).

[4]Under conformal mapping (3.32) this state is transformed into a homogeneous polynomial of degree h, $\widetilde{\Psi}(w_1,w_2) = (w_1-w_2)^h$.

According to (2.15), $B(u)$ is a polynomial in u of degree $2N-1$ with operator-valued coefficients, $B(u) = 2i(-1)^{N-1}S_- u^{2N-1} + \cdots$. In addition, it follows from (2.19) that $B(-i/2) = 0$ and, moreover, $B(u)/(2u+i)$ is an even function of u. This suggests to remove the "kinematic" zero of $B(u)$ and define the operator

$$\widehat{B}(u) = \frac{B(u)}{2u+i} = (-1)^{N-1} i S_- \left(u^{2N-2} + \widehat{b}_2\, u^{2N-4} + \cdots + \widehat{b}_N \right), \qquad (4.4)$$

with $\widehat{b}_2, \ldots, \widehat{b}_N$ being some (commuting) operators. Since $B(u) = b(-u)a(u) - a(-u)b(u)$ (see eq. (3.13)), one finds using $(a(u))^\dagger = a(u^*)$ and $(b(u))^\dagger = b(u^*)$ that $(B(u))^\dagger = -B(-u^*)$, or equivalently $(\widehat{B}(u))^\dagger = \widehat{B}(u^*)$. Thus, $\widehat{B}(u)$ is hermitian operator for real u.

Following Sklyanin [7], we identify the eigenstates of the operator $\widehat{B}(u)$ as the kernel of the transition operator to the SoV representation

$$\widehat{B}(u)\, U_{p,\boldsymbol{x}}(z_1, \ldots, z_N) = (-1)^{N-1}\, p\, (u^2 - x_1^2) \cdots (u^2 - x_{N-1}^2)\, U_{p,\boldsymbol{x}}(z_1, \ldots, z_N)\,. \qquad (4.5)$$

According to (2.17), $[\widehat{B}(u), \widehat{B}(v)] = 0$ and, therefore, $U_{p,\boldsymbol{x}}(z_1, \ldots, z_N)$ does not depend on the spectral parameter u. Due to (4.4), the corresponding eigenvalues are real polynomials in u^2 of degree $N-1$. They can be parameterized by the total momentum p and by the set of parameters $\boldsymbol{x} = (x_1, \ldots, x_{N-1})$ which are identified as the separated variables. Hermiticity of the operator $\widehat{B}(u)$ implies that x_k^2 can be either real, or can appear in complex conjugated pairs, $x_k^2 = (x_j^2)^*$. We will argue in the next section that the separated variables satisfy a much stronger condition $x_k^2 > 0$, which together with the symmetry of (4.5) under $x_k \to -x_k$ allows one to assign to the separated variables $\boldsymbol{x}$ *real positive* values. This follows from the requirement that $U_{p,\boldsymbol{x}}(z_1, \ldots, z_N)$ have to be the eigenstates of the self-adjoint operator $\widehat{B}(u)$ and, therefore, they have to fulfill the completeness condition

$$\int_0^\infty dp \int_{\mathbb{R}_+^{N-1}} d^{N-1}\boldsymbol{x}\, \mu(\boldsymbol{x})\, (U_{p,\boldsymbol{x}}(w_1, \ldots, w_N))^*\, U_{p,\boldsymbol{x}}(z_1, \ldots, z_N) = \prod_{n=1}^{N} \mathbb{K}(z_n|\bar{w}_n) \qquad (4.6)$$

where $\mathbb{K}(z|\bar{w}) = \mathrm{e}^{i\pi s}(z-\bar{w})^{-2s}$ is the kernel of the identity operator (see eq. (A.4)).

The diagonal element $D(\pm x_k)$ of the monodromy matrix (2.15) acts on $U_{p,\boldsymbol{x}}(w_1, \ldots, w_N)$ as a shift operator

$$D(\pm x_k) U_{p,\boldsymbol{x}}(z_1, \ldots, z_N) = \delta(\pm x_k) U_{p,\boldsymbol{x}\pm i\boldsymbol{e}_k}(z_1, \ldots, z_N)\,. \qquad (4.7)$$

Indeed, taking $v = \pm x_k$ in the second fundamental relation (2.17) and applying its both sides to $U_{p,\boldsymbol{x}}(z_1, \ldots, z_N)$, one arrives at (4.7). The scalar factor $\delta(x_k)$ depends on the normalization of $U_{p,\boldsymbol{x}}(z_1, \ldots, z_N)$. Applying $U_{p,\boldsymbol{x}}$ to the both sides of (2.16) and taking $u = x_k$ one finds that $\delta(x_k)$ satisfies the relation

$$\delta(x_k)\delta(-x_k - i) = [(x_k + is)(x_k + i(1-s))]^{2N}\,. \qquad (4.8)$$

In (4.7) it was tacitly assumed that the function $U_{p,\boldsymbol{x}}(z_1, \ldots, z_N)$ can be continued to complex $\boldsymbol{x}$. Notice that $U_{p,\boldsymbol{x}\pm i\boldsymbol{e}_k}(z_1, \ldots, z_N)$ is not the eigenfunction of the operator $\widehat{B}(u)$ even though it satisfies the differential equation (4.5).

4.1 Transition kernel

Solving the spectral problem (4.5), we follow the approach developed in ref. [13] in application to the closed spin chain. To begin with, we notice that the differential equation (4.5) is equivalent to the system of N equations

$$iS_-\,U_{p,\boldsymbol{x}}(z) = p\,U_{p,\boldsymbol{x}}(z)\,, \qquad \widehat{B}(\pm x_k)\,U_{p,\boldsymbol{x}}(z) = 0\,, \qquad (k = 1,\dots N-1)\,. \tag{4.9}$$

Let us consider the second relation and compare it with a similar relation (3.21) for $u = \pm x_k$. Sending the gauge parameter w_1 in (3.21) to infinity and taking into account (3.12) one finds that $B(\pm x_k)$ annihilates the following function

$$\Lambda_x(z_1,\dots,z_N|\bar w_2\dots,\bar w_N) = \lim_{\bar w_1\to\infty} \bar w_1^{2s}\,Q_x(z_1,\dots,z_N|\bar w_1,\dots,\bar w_N)\,. \tag{4.10}$$

Here the additional prefactor is inserted to make the limit finite

$$\begin{aligned}\Lambda_u(z_1,\dots,z_N|\bar w_2\dots,\bar w_N) \,=\, & \mathrm{e}^{i\pi s(2N-1)}\int \mathcal{D}y_2\cdots\mathcal{D}y_N\,(y_N-\bar w_N)^{-\beta_u}(z_N-\bar w_N)^{-\alpha_u}\,\times\\ & \times\prod_{k=2}^{N}(z_{k-1}-\bar y_k)^{-\alpha_u}(z_k-\bar y_k)^{-\beta_u}\,\times\\ & \times\prod_{n=2}^{N-1}(y_n-\bar w_n)^{-\beta_u}(y_{n+1}-\bar w_n)^{-\alpha_u}\,.\end{aligned} \tag{4.11}$$

As before, it is convenient to represent this expression as the Feynman diagram shown in figure 3. It differs from the Feynman diagram for the $\mathbb{Q}$-operator (see figure 1) in that two lines attached to the vertex $\bar w_1$ are removed.

By the construction, the Λ-function satisfies the relation

$$\widehat{B}(\pm x_k)\,\Lambda_{x_k}(z_1,\dots,z_N|\bar w_2\dots,\bar w_N) = 0\,. \tag{4.12}$$

Let us introduce the integral operator $\Lambda_N(x)$ with the kernel given by (4.11). It maps a function of $N-1$ variables $\Psi_{N-1}(w_2,\dots,w_N)$ into a function of N variables $\Psi_N(z_1,\dots,z_N)$

$$\begin{aligned}\Psi_N(z_1,\dots,z_N) \,&=\, [\Lambda_N(u)\,\Psi_{N-1}](z_1,\dots,z_N)\\ &= \int\mathcal{D}w_2\cdots\int\mathcal{D}w_N\,\Lambda_u(z_1,\dots,z_N|\bar w_2\dots,\bar w_N)\Psi_{N-1}(w_2,\dots,w_N)\,,\end{aligned} \tag{4.13}$$

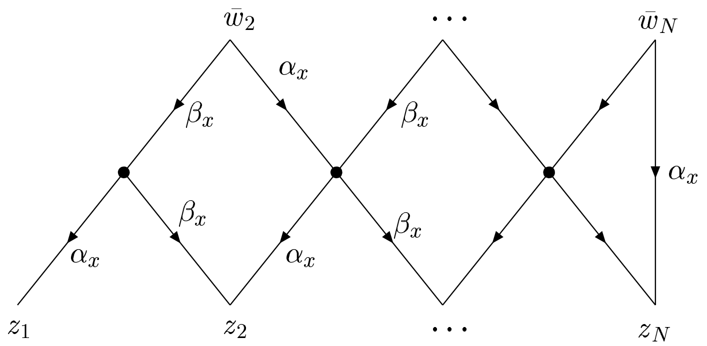

Figure 3: The diagrammatical representation of the function $\Lambda_x(z_1,\dots,z_N|\bar w_2,\dots,\bar w_N)$.

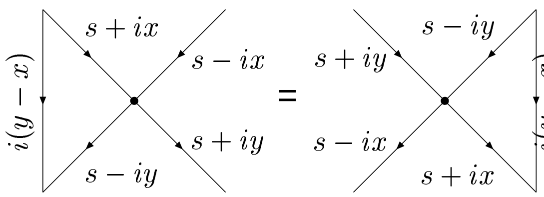

Figure 4: Permutation identity.

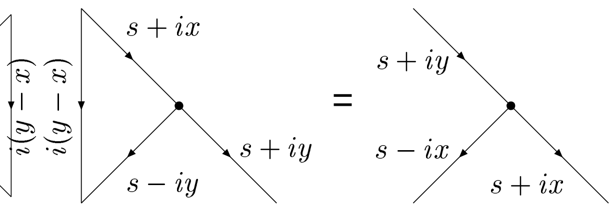

Figure 5: Special case of the permutation identity. It is obtained from figure 4 by sending one of the external points to infinity.

The operator $\Lambda_N(x)$ defined in this way has a number of remarkable properties:

- Parity:

$$\Lambda_N(x) = \Lambda_N(-x) \tag{4.14}$$

- Commutativity:

$$\Lambda_N(x_1)\Lambda_{N-1}(x_2) = \Lambda_N(x_2)\Lambda_{N-1}(x_1) \tag{4.15}$$

- Baxter relation:

$$\widehat{t}_N(x)\,\Lambda_N(x) = \Delta_+(x)\,\Lambda_N(x+i) + \Delta_-(x)\,\Lambda_N(x-i) \tag{4.16}$$

- Exchange relation:

$$\Lambda_N^\dagger(x)\Lambda_N(y) = \varphi(x,y)\cdot\Lambda_{N-1}(y)\Lambda_{N-1}^\dagger(x)\,, \tag{4.17}$$

where $x \neq y$ and the scalar function $\varphi(x,y) = \varphi(y,x)$ is defined as

$$\varphi(x,y) = \mathrm{e}^{4i\pi s}\; a(\alpha_x,\alpha_y)\,a(\beta_x,\beta_y)\,a(\beta_x,\alpha_y)\,a(\alpha_x,\beta_y) \tag{4.18}$$

with $\alpha_x = s - ix$ and $\beta_x = s + ix$,

$$a(\alpha,\beta) = \mathrm{e}^{-i\pi s}\,\frac{\Gamma(\alpha+\beta-2s)\Gamma(2s)}{\Gamma(\alpha)\Gamma(\beta)}\,. \tag{4.19}$$

The following comments are in order.

Eq. (4.14) follows from the parity property of the Baxter $\mathbb{Q}$-operator, eq. (3.22). The proof of (4.15) can be performed diagrammatically with a help of the permutation identities (see figures 4 and 5). It goes along the same lines as the proof of commutativity property for the $\mathbb{Q}$-operator presented at the end of section 3.2. Eq. (4.16) follows immediately from (4.10) and (3.3). The proof of the exchange relation (4.17) is illustrated in figure 6. The product $\Lambda_N^\dagger(x)\Lambda_N(y)$ corresponds to the left diagram in figure 6. The left-most vertex in this diagram can be integrated out with a help of the chain relation (see figure 9) producing a single line with the index $\alpha_x+\beta_y-2s = -i(x-y)$. Then, one moves this line horizontally to the right of the diagram by applying the permutation identity (see figure 4). Repeating the same steps for the resulting diagram one finally arrives at the right diagram in figure 6 with the additional prefactor (4.18).

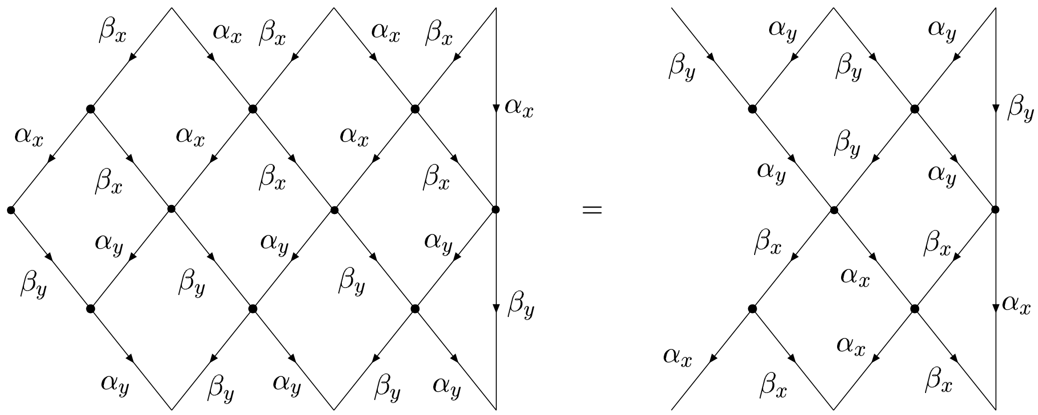

Figure 6: Exchange relation. Here $\alpha_x = s - ix$ and $\beta_x = s + ix$.

Taking into account the properties of the Λ-operator, it becomes straightforward to write a general solution to the system (4.9)

$$U_{p,\boldsymbol{x}}(z_1,\ldots,z_N) = p^{Ns-1/2}\int \mathcal{D}w_N\; \mathrm{e}^{ip\,w_N}\; U_{\boldsymbol{x}}(\vec{z}\,;\bar{w}_N)\,, \tag{4.20}$$

where $U_{\boldsymbol{x}}(\vec{z}\,;\bar{w}_N)$ is factorized into the product of $N-1$ operators

$$U_{\boldsymbol{x}}(\vec{z}\,;\bar{w}_N) = \left[\Lambda_N(x_1)\,\Lambda_{N-1}(x_2)\cdots\Lambda_2(x_{N-1})\right](z_1,\ldots,z_N|\bar{w}_N)\,, \tag{4.21}$$

with $\vec{z} = (z_1,\ldots,z_N)$ and the additional factor $p^{Ns-1/2}$ introduced in (4.20) for the later convenience. Indeed, the first relation in (4.9) is satisfied due to invariance of (4.21) under translations $z_k \to z_k+\epsilon$ and $\bar{w}_N \to \bar{w}_N+\epsilon$ with ϵ real. It follows from (4.14) and (4.15) that $U_{\boldsymbol{x}}(\vec{z}\,;\bar{w}_N)$ is an even symmetric function of $x_1,\ldots,x_{N-1}$. Since $\widehat{B}(\pm x_1)\,U_{\boldsymbol{x}}(\vec{z}\,;\bar{w}_N) = 0$ by virtue of (4.21) and (4.12), the second relation in (4.9) is fulfilled for arbitrary k. Notice that the kernel $U_{\boldsymbol{x}}(\vec{z}\,;\bar{w}_N)$ satisfies a multi-dimensional Baxter relation

$$\widehat{t}_N(x_k)\,U_{\boldsymbol{x}}(\vec{z}\,;\bar{w}_N) = \Delta_+(x_k)U_{\boldsymbol{x}+i\boldsymbol{e}_k}(\vec{z}\,;\bar{w}_N) + \Delta_-(x_k)U_{\boldsymbol{x}-i\boldsymbol{e}_k}(\vec{z}\,;\bar{w}_N)\,, \tag{4.22}$$

where $\boldsymbol{e}_k$ denotes a unit basis vector in the $\boldsymbol{x}$-space, $\boldsymbol{x} = \sum_k x_k\boldsymbol{e}_k$. Eq. (4.22) follows from the similar property of the Λ-operator, eq. (4.16), and the symmetry of the kernel under permutations of x-variables.

Eqs. (4.20) and (4.21) define the transition kernel $U_{p,\boldsymbol{x}}(z_1,\ldots,z_N)$ to the SoV representation for the homogeneous open spin chain. Remarkably enough, these expressions have the same form as for the closed SL(2) spin chain [13]. The only difference between the two cases is in the definition of the Λ-operator. Diagrammatical representation for the transition kernel (4.21) is shown in figure 7. The corresponding Feynman diagram has a pyramidal form which reflects the structure of the kernel (4.21). It consists of $(N-1)$-rows with each row representing a single Λ-operator.

It remains to verify that the kernel (4.20) satisfies for real $\boldsymbol{x}$ the completeness condition (4.6). As we will show in section 5, the transformation to the SoV representation for $N=2$ open spin chain coincides with the Fourier-Jacobi transform (see, e.g. ref. [19]). Then,

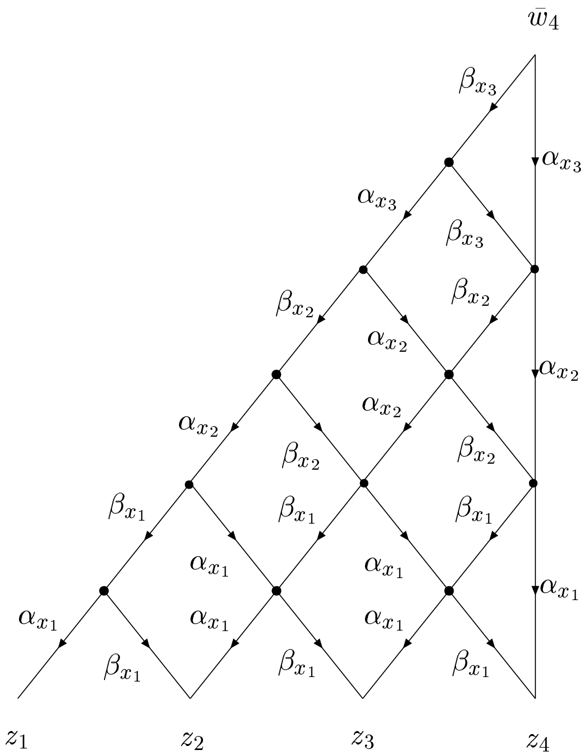

Figure 7: The diagrammatic representation of the kernel $U_{\boldsymbol{x}}(\vec{z}, w_N)$ for $N = 4$.

reality condition for the separated variable x and completeness condition for $U_{p,x}(z_1, z_2)$ follow immediately from the properties of the Fourier-Jacobi transform. For $N \geq 3$ some arguments will be presented in the next subsection.

4.2 Integration measure

Let us demonstrate that the transition kernel defined in eqs. (4.20) and (4.21) satisfies the orthogonality condition

$$\begin{aligned}\langle p', \boldsymbol{x}' | p, \boldsymbol{x} \rangle &= \int \mathcal{D}^N z \, U_{p,\boldsymbol{x}}(z_1, \ldots, z_N) \big(U_{p',\boldsymbol{x}'}(z_1, \ldots, z_N) \big)^* \\ &= \delta(p - p') \left\{ \delta(\boldsymbol{x} - \boldsymbol{x}') + \cdots \right\} \frac{\mu^{-1}(\boldsymbol{x})}{(N-1)!} \,, \end{aligned} \tag{4.23}$$

and calculate the integration measure $\mu(\boldsymbol{x})$. Here $\delta(\boldsymbol{x} - \boldsymbol{x}') \equiv \prod_{k=1}^{N-1} \delta(x_k - x_k')$ and $\boldsymbol{x} = (x_1, \ldots, x_{N-1})$ take positive real values, $x_k > 0$. Ellipses denote the terms with all possible permutations inside the set $\boldsymbol{x} = (x_1, \ldots, x_{N-1})$.

The calculation of (4.23) repeats similar analysis for the closed spin chain described at length in ref. [13]. Substitution of (4.20) into (4.23) yields

$$\langle p', \boldsymbol{x}' | p, \boldsymbol{x} \rangle = (pp')^{Ns-1/2} \int \mathcal{D} w_N \, \mathrm{e}^{ip\, w_N} \int \mathcal{D} w_N' \, \mathrm{e}^{-ip'\, \bar{w}_N'} \langle w_N', \boldsymbol{x}' | w_N, \boldsymbol{x} \rangle \,, \tag{4.24}$$

where the notation was introduced for the ket-vector $\langle z_1, \ldots, z_N | w_N, \boldsymbol{x}\rangle = U_{\boldsymbol{x}}(\vec{z}\,; \bar{w}_N)$, or equivalently

$$|w_N, \boldsymbol{x}\rangle = \Lambda_N(x_1)\Lambda_{N-1}(x_2)\cdots\Lambda_2(x_{N-1})|w_N\rangle\,, \tag{4.25}$$

with $|w_N\rangle$ being a "single-particle state". We recall that the operator $\Lambda_k(u)$ maps $(k-1)$-particle state into k-particle one, so that a composition of the Λ-operators in (4.25) produces the N-particle state. Calculating the scalar product $\langle w_N', \boldsymbol{x}'|w_N, \boldsymbol{x}\rangle$ one applies systematically the exchange relation (4.17) and obtains

$$\langle w_N', \boldsymbol{x}'|w_N, \boldsymbol{x}\rangle = c(\boldsymbol{x}, \boldsymbol{x}')\langle w_N'|\Big(\Lambda_2^\dagger(x_{N-1}')\Lambda_2(x_1)\Big)\cdots\Big(\Lambda_2^\dagger(x_1')\Lambda_2(x_{N-1})\Big)|w_N\rangle\,, \tag{4.26}$$

where $c(\boldsymbol{x}, \boldsymbol{x}') = \prod_{\substack{1\le j,\,k\le N-2\\ j+k\le N-1}} \varphi(x_j, x_k')$. Notice that the exchange relation (4.17) holds only for $x \neq y$. Therefore, calculating (4.26) we have tacitly assumed that $x_j \neq x_k'$ for $j+k \le N-1$, or equivalently that all factors $\varphi(x_j, x_k')$ are finite. For $N \ge 3$ the matrix element entering (4.26) can be represented as follows

$$\int \mathcal{D}w_1 \cdots \int \mathcal{D}w_{N-2}\langle w_N', x_{N-1}'|w_1, x_1\rangle\langle w_1, x_{N-2}'|w_2, x_2\rangle\cdots\langle w_{N-2}, x_1'|w_N, x_{N-1}\rangle\,, \tag{4.27}$$

where $\langle w', x'|w, x\rangle = [\Lambda_2^\dagger(x')\,\Lambda_2(x)](w'; w)$. Thus, the calculation of the scalar product (4.24) for arbitrary N is reduced to the calculation of $\langle w', x'|w, x\rangle$ at $N=2$. Given that the separated variables at $N=2$ take real positive values we deduce from (4.26) that the same holds true for arbitrary N.

To calculate the scalar product at $N=2$ we apply the diagrammatical approach of ref. [13] and represent the matrix element $\langle w', x'|w, x\rangle$ as the Feynman diagram shown in figure 8. One expects from (4.23) that $\langle w', x'|w, x\rangle \sim \delta(x-x')$, so that the scalar product $\langle w', x'|w, x\rangle$ should be understood as a distribution. To find its explicit form we regularize the corresponding Feynman integral by introducing a small parameter ϵ and shifting the indices of two lines as indicated in figure 8. Under such regularization, the Feynman integral remains finite at $x = x'$ and it can be calculated exactly with a help of the chain relation and permutation identity (see figures 9 and 4). The calculation is straightforward and some details can be found in ref. [13]. Going over to the momentum representation and taking the limit $\epsilon \to 0$, one finds (for $x\,, x' > 0$)

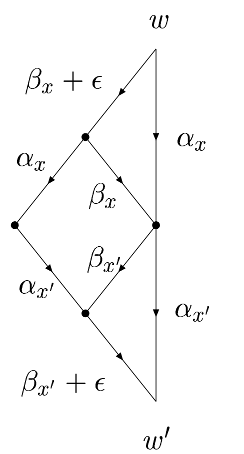

Figure 8: The scalar product $\langle w', x'|w, x\rangle$ for $N=2$.

$$\int \mathcal{D}w\ \mathrm{e}^{ipw}\,\langle w', x'|w, x\rangle = 2\pi\ \mathrm{e}^{ipw'}\ p^{-2s}\,\Gamma^5(2s)\,\frac{|\Gamma(i(x+x'))|^2}{|\Gamma(s+ix)\Gamma(s+ix')|^4}\,\delta(x-x')\,. \tag{4.28}$$

Substituting (4.26) and (4.27) into (4.24) and taking into account (4.28) one obtains after some algebra

$$\langle p', \boldsymbol{x}'|p, \boldsymbol{x}\rangle = (2\pi)^{N-1}\delta(p-p')\prod_{k=1}^{N-1}\delta(x_k - x_{N-k}') \times$$

$$\times\Gamma^N(2s)\prod_{k=1}^{N-1}\left[\frac{\Gamma(s-ix_k)\Gamma(s+ix_k)}{\Gamma(2s)}\right]^{-2N}\times \tag{4.29}$$
$$\times\left(\prod_{1\le j\le k\le N-1}\frac{x_k+x_j}{\pi}\sinh\pi(x_k+x_j)\prod_{1\le j<k\le N-1}\frac{x_k-x_j}{\pi}\sinh\pi(x_k-x_j)\right)^{-1}.$$

We recall that the calculation was performed under assumption that $x_j\neq x'_k$ for $j+k\le N-1$. Since the kernel $U_{p,\boldsymbol{x}}$ is a symmetric function of $\boldsymbol{x}$, $\langle p',\boldsymbol{x}'|p,\boldsymbol{x}\rangle$ should possess the same property. This allows one to relax the above assumption and replace the product of delta-functions $\prod_{k=1}^{N-1}\delta(x_k-x'_{N-k})$ in the r.h.s. of (4.29) by the sum $\sum_{\mathcal{S}}\delta(\boldsymbol{x}-\mathcal{S}\boldsymbol{x}')$ over all permutations inside the set $\boldsymbol{x}'=\left(x'_1,\dots,x'_{N-1}\right)$.

Matching (4.29) into (4.23), one finds the expression for the integration measure in the SoV representation

$$\mu(\boldsymbol{x})=\frac{\Gamma^{-N}(2s)}{(N-1)!(2\pi)^{N-1}}\prod_{k=1}^{N-1}\left[\frac{\Gamma(s-ix_k)\Gamma(s+ix_k)}{\Gamma(2s)}\right]^{2N}\times \tag{4.30}$$
$$\times\prod_{1\le j<k\le N-1}\frac{x_k^2-x_j^2}{2\pi^2}\left[\cosh(2\pi x_k)-\cosh(2\pi x_j)\right]\prod_{k=1}^{N-1}\frac{2x_k}{\pi}\sinh(2\pi x_k)\,.$$

This expression has the following properties. As expected, $\mu(\boldsymbol{x})$ is an even function of the separated variables. It takes nonnegative values for real $\boldsymbol{x}=(x_1,\dots,x_{N-1})$ and vanishes on the hyperplanes $x_j=x_k$.

After analytical continuation to complex $\boldsymbol{x}$, the measure $\mu(\boldsymbol{x})$ becomes a meromorphic function of x_k $(k=1,\dots,N-1)$ with poles of the order $2N$ located along the imaginary axis at $x_k=\pm i(s+n)$ with $n\in\mathbb{N}$. The measure decreases exponentially fast when one of the separated variables, say x_k, goes to infinity along the real axis

$$\mu(\boldsymbol{x})\sim \mathrm{e}^{-2\pi|x_k|}\,x_k^{2Ns-3}\,. \tag{4.31}$$

One verifies that the measure (4.30) satisfies the functional relation

$$\frac{\mu(\boldsymbol{x}+i\boldsymbol{e}_k)}{\mu(\boldsymbol{x})}=\frac{x_k+i}{x_k}\left(\frac{x_k+is}{x_k+i(1-s)}\right)^{2N}\prod_{j\neq k}\frac{x_k-x_j+i}{x_k-x_j}\frac{x_k+x_j+i}{x_k+x_j}\,, \tag{4.32}$$

with $\boldsymbol{e}_k$ defined in (4.22).

It is instructive to compare (4.30) with a similar expression for the integration measure for the *closed* spin chain [13]

$$\mu_{\rm cl}(\boldsymbol{x})=\prod_{\substack{j,k=1\\ j<k}}^{N-1}(x_k-x_j)\,\sinh(\pi(x_k-x_j))\prod_{k=1}^{N-1}\left[\Gamma(s+ix_k)\Gamma(s-ix_k)\right]^N. \tag{4.33}$$

One observes that $\mu_{\rm cl}(\boldsymbol{x})$ enters as a factor into the expression for $\mu(\boldsymbol{x})$, eq. (4.30).

4.3 Eigenfunctions in the SoV representation

The eigenfunctions $\Psi_{\boldsymbol{q},p}(z_1,\ldots,z_N)$ are orthogonal to each other for different sets of quantum numbers with respect to the SL(2) scalar product (2.5). In the SoV representation the same condition looks as follows

$$\langle\Psi_{\boldsymbol{q'},p'}|\Psi_{\boldsymbol{q},p}\rangle = \langle\Phi_{\boldsymbol{q'}}|\Phi_{\boldsymbol{q}}\rangle_{\rm SoV}\,\delta(p-p') = \delta(p-p')\,\delta_{\boldsymbol{q},\boldsymbol{q'}}\,, \tag{4.34}$$

where the scalar product in the SoV representation is given by

$$\langle\Phi_{\boldsymbol{q'}}|\Phi_{\boldsymbol{q}}\rangle_{\rm SoV} = \int_{\mathbb{R}^+_{N-1}} d^{N-1}\boldsymbol{x}\mu(\boldsymbol{x})\big(\Phi_{\boldsymbol{q'}}(x_1,\ldots,x_{N-1})\big)^*\Phi_{\boldsymbol{q}}(x_1,\ldots,x_{N-1})\,. \tag{4.35}$$

We recall that the momentum p takes real positive values whereas the integrals of motion $\boldsymbol{q}=(q_2,\ldots,q_N)$ have a discrete spectrum.

To define the eigenfunction in the SoV representation, $\Phi_{\boldsymbol{q}}(\boldsymbol{x})$, we substitute $\Psi_{\boldsymbol{q},p}(z_1,\ldots,z_N)$ in (2.25) by its integral representation (4.1). Following the standard procedure [7] and making use of eqs. (4.22) and (4.32) one can show that $\Phi_{\boldsymbol{q}}(\boldsymbol{x})$ satisfies the $(N-1)$-dimensional Baxter equation

$$t_N(x_k)\,\Phi_{\boldsymbol{q}}(\boldsymbol{x}) = \Delta_+(x_k)\Phi_{\boldsymbol{q}}(\boldsymbol{x}+i\boldsymbol{e}_k) + \Delta_-(x_k)\Phi_{\boldsymbol{q}}(\boldsymbol{x}-i\boldsymbol{e}_k)\,, \tag{4.36}$$

where $t_N(x_k)$ is the eigenvalue of the transfer matrix, eq. (2.25). As before, to solve this equation one has to specify additional conditions for $\Phi_{\boldsymbol{q}}(\boldsymbol{x})$.

Using (4.3) one can show that $\Phi_{\boldsymbol{q}}(\boldsymbol{x})$ is a polynomial in $\boldsymbol{x}=(x_1,\ldots,x_{N-1})$. The proof goes along the same lines as analysis of analytical properties of the Baxter $\mathbb{Q}$-operator in section 3.3. Namely, substituting the expression for the kernel (4.20) into (4.3) and applying (A.5), one can express the r.h.s. of (4.3) as a nested contour integral. Analytical properties of the function $\Phi_{\boldsymbol{q}}(\boldsymbol{x})$ are in the one-to-one correspondence with the properties of this integral.

It is easy to see that polynomial solutions to (4.36) can be represented in the factorized form

$$\Phi_{\boldsymbol{q}}(\boldsymbol{x}) = c_{\boldsymbol{q}}\,Q_{\boldsymbol{q}}(x_1)\cdots Q_{\boldsymbol{q}}(x_{N-1})\,, \tag{4.37}$$

where $Q_{\boldsymbol{q}}(x)$ is the eigenvalue of the Baxter $\mathbb{Q}$-operator, eq. (3.33), and the coefficient $c_{\boldsymbol{q}}$ is fixed by the normalization condition (4.34). Substituting (4.37) into (4.35) and taking into account that $Q_{\boldsymbol{q}}(x)$ is a real function, eq. (3.33), we find that the solutions to the Baxter equation satisfy the orthogonality condition

$$\int_{\mathbb{R}^{N-1}_+} d^{N-1}\boldsymbol{x}\,\mu(\boldsymbol{x})\prod_{k=1}^{N-1} Q_{\boldsymbol{q'}}(x_k)\,Q_{\boldsymbol{q}}(x_k) \sim \delta_{\boldsymbol{q},\boldsymbol{q'}}\,, \tag{4.38}$$

with the measure given by (4.30).

Thus, having determined the eigenvalues of Baxter operator $Q_{\boldsymbol{q}}(x)$ one would be able to restore both the energy and the corresponding eigenfunction, eqs. (3.38) and (4.1), respectively. The solutions to the Baxter equation for the SL$(2,\mathbb{R})$ open spin chain have been studied in [4]. It turns out that the ground state $\Omega_p(\vec{z})$ of the model can be found exactly. This state has the total SL(2) spin $h=0$, eq. (3.31), and has the form[5]

$$\Omega_p(\vec{z}) = \frac{p^{2s-1}}{\Gamma(2s)} \int \mathcal{D}w\, \mathrm{e}^{ipw} \prod_{k=1}^{N} (z_k - \bar{w})^{-2s} \,. \tag{4.39}$$

Indeed, $\Omega_p(\vec{z})$ diagonalizes simultaneously the operators of two-particle spins, $(J_{n,n+1} - 2s)\Omega_p(\vec{z}) = 0$, eq. (2.2), and leads the energy $E_{\boldsymbol{q}} = 0$. It is interesting to notice that $\Omega_p(\vec{z})$ is also the ground state of the homogeneous SL$(2,\mathbb{R})$ *closed* spin chain [13].

The state (4.39) diagonalizes the Baxter operator $\mathbb{Q}(u)|\Omega_p\rangle = |\Omega_p\rangle$, or equivalently $Q_{\boldsymbol{q}}(u) = 1$. As a consequence, it admits the following integral representation (see ref. [13])[6]

$$\Omega_p(\vec{z}) = p^{Ns-1/2}\, \mathrm{e}^{-i\pi s(2N-1)} \int_{\mathbb{R}_+^{N-1}} d^{N-1}\boldsymbol{x}\, \mu(\boldsymbol{x}) U_{p,\boldsymbol{x}}(\vec{z}) \,. \tag{4.40}$$

Let us calculate the SL(2) scalar product $\langle \Omega_p | \Omega_{p'} \rangle$ and use two different expressions for $\Omega_p(\vec{z})$, eqs. (4.39) and (4.40). Equating the two expressions, one finds that the integration measure (4.30) satisfies the normalization condition

$$\int_{\mathbb{R}_+^{N-1}} d\boldsymbol{x}\, \mu(\boldsymbol{x}) = \frac{1}{\Gamma(2Ns)} \,. \tag{4.41}$$

The relation (4.40) allows one to establish the equivalence between the SoV and the ABA methods for the SL$(2,\mathbb{R})$ open spin chain. Let $\lambda_1, \dots, \lambda_h$ be the Bethe roots, or equivalently zeros of the polynomial $Q_{\boldsymbol{q}}(x)$, eq. (3.33). Applying $\widehat{B}(\lambda_1)\cdots\widehat{B}(\lambda_h)$ to the both sides of (4.40) and taking into account (4.5) we obtain

$$\Psi_{p,\boldsymbol{q}}(\vec{z}) = \widehat{B}(\lambda_1)\cdots\widehat{B}(\lambda_h)\, \Omega_p(\vec{z}) = c(p) \int_{\mathbb{R}_+^{N-1}} d\boldsymbol{x}\, \mu(\boldsymbol{x})\, U_{p,x}(\vec{z}) \prod_{k=1}^{N-1} Q_{\boldsymbol{q}}(x_k) \,, \tag{4.42}$$

where $c(p)$ is the normalization factor. In (4.42), the first relation coincides with the ABA representation for the eigenstate of the model while the second one defines the same eigenstate in the SoV representation.

The explicit form of the eigenfunctions in the SoV representation (4.37) suggests that there exists a relation between the Baxter $\mathbb{Q}$-operator and the transition kernel $U_{p,\boldsymbol{x}}(\vec{z})$ [20]. In the case of the closed spin chain it has been established in ref. [13]. It turns out that this relation is universal and it also holds for the open spin chain

$$U_{p,\boldsymbol{x}}(\vec{z}) = [\mathbb{Q}(x_1)\, \dots \mathbb{Q}(x_{N-1})\, \Theta_p\,](z_1, \dots, z_N) \,. \tag{4.43}$$

Here $\Theta_p(\vec{z})$ is a certain $\boldsymbol{x}$-independent function of $\vec{z}$, which does not belong to the quantum space of the model.[7] Since $\mathbb{Q}(is) = \mathbb{K}$, the function $\Theta_p(\vec{z})$ is equal to $U_{p,\boldsymbol{x}}(\vec{z})$ for special

[5] In the terminology of the Algebraic Bethe Ansatz, $\Omega_p(\vec{z})$ is a pseudovacuum state.

[6] Notice that this relation takes the same form both for the open and closed spins chain whereas the expressions for the integration measure and the transition kernel to the SoV representation are different in the two cases.

[7] In ref. [13], the function $\Theta_p(\vec{z})$ was defined as a limiting case of the state $|\Omega_{\bar{w}_0,\bar{w}_N}\rangle$ belonging to the Hilbert space of the model.

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values of the x-variables, $x_1 = \cdots = x_{N-1} = is$ that we denote as $U_{p,is}(\vec{z})$. The expression for $U_{p,is}(\vec{z})$ can be easily obtained from diagrammatic representation of the kernel (see figure 7)

$$\Theta_p(\vec{z}) = U_{p,is}(\vec{z}) = p^{Ns-1/2}\, \mathrm{e}^{i\pi s(N-1)}\, \mathrm{e}^{ipz_N}\ . \tag{4.44}$$

Notice that $\Theta_p(\vec{z})$ depends only on a single variable z_N and, therefore, it is not normalizable with respect to the SL(2) scalar product (2.5).

The proof of (4.43) can be performed diagrammatically and it repeats similar analysis in ref. [13]. Another way to verify (4.43) is to use the following identities

$$\begin{aligned} \int \mathcal{D}^N w\, Q_u(\vec{z}|\vec{w})\, \mathrm{e}^{ipw_N} &= \mathrm{e}^{-i\pi s} \int \mathcal{D} w_N\, \Lambda_u(z_{N-1}, z_N|\bar{w}_N)\, \mathrm{e}^{ipw_N} \\ \int \mathcal{D}^N w\, Q_u(\vec{z}|\vec{w}) \Lambda_v(w_k, \ldots, w_N|\bar{y}_{k+1}, \ldots, \bar{y}_N) &= \mathrm{e}^{-i\pi s}[\Lambda_{k+1}(u)\Lambda_k(v)] \times \\ &\quad \times (z_{k-1}, \ldots, z_N; \bar{y}_{k+1}, \ldots, \bar{y}_N)\,, \end{aligned} \tag{4.45}$$

with $\vec{z} = (z_1, \ldots, z_N)$, $\vec{w} = (\bar{w}_1, \ldots, \bar{w}_N)$ and $\mathcal{D}^N w = \prod_{n=1}^N \mathcal{D} w_n$. We recall that the Λ-operator was defined in eqs. (4.13) and (4.11). To derive (4.45) one substitutes the expression for $Q_u(\vec{z}|\vec{w})$, eq. (3.16), and integrates over "free" vertices $w_1, \ldots, w_{k-1}$ with a help of the identity (A.5) for $\Psi(w) = 1$. Eqs. (4.45) can be rewritten symbolically as

$$\mathbb{Q}(u)|\Theta_p\rangle = \mathrm{e}^{-i\pi s}\, \Lambda_2(u)|\Theta_p\rangle\,, \qquad \mathbb{Q}(u)\Lambda_k(v) = \mathrm{e}^{-i\pi s}\, \Lambda_{k+1}(u)\Lambda_k(v)\,. \tag{4.46}$$

Applying these relations one verifies that (4.43) coincides with (4.20) and (4.21).

5. Relation to the Wilson polynomials

In this section, we consider the open spin chain with $N = 2$ sites. We will demonstrate that in that case the eigenvalues of the Baxter $\mathbb{Q}$-operator are given by the Wilson polynomials [21] and the unitary transformation to the SoV representation coincides with the Fourier-Jacobi transformation [19].

At $N = 2$ the hamiltonian of the open spin chain (2.1) equals $\mathcal{H}_2 = H_{12} = 2[\psi(J_{12}) - \psi(2s)]$. Its eigenstates are uniquely fixed by the values of the momentum p and the total SL(2) spin h, eq. (3.31) and are given by

$$\Psi_{p,h}(z_1, z_2) = \frac{p^{2s-1}}{\Gamma(2s)} \int \mathcal{D} w\ \mathrm{e}^{ipw}\, \frac{(z_1 - z_2)^h}{(z_1 - \bar{w})^{2s+h}(z_1 - \bar{w})^{2s+h}}\,. \tag{5.1}$$

Substituting (5.1) into (3.28), one finds the eigenvalues of the $N = 2$ Baxter operator after some algebra as[8]

$$Q_h(u) = {}_4F_3\left(\left. \begin{matrix} -N, N+4s-1, s+iu, s-iu \\ 2s, 2s, 2s \end{matrix} \right| 1 \right) = \left[\frac{\Gamma(2s)}{\Gamma(2s+N)} \right]^3 W_N(u^2, s, s, s, s)\,, \tag{5.2}$$

[8]As was explained in section 3.3, due to the SL(2) invariance of the Baxter operator, one can calculate $Q_h(u)$ by substituting $\Psi(z_1, z_2) = (z_1 - z_2)^h$ into (3.28).

where $W_N(u^2)$ is the Wilson polynomial [21]. It is interesting to note that the solution of the $N = 2$ Baxter equation for the closed spin chain is given by the continuous Hahn polynomials (see e.g. ref. [13]). The polynomials $Q_h(x)$ are orthogonal on the half-axis $x > 0$ with respect to the scalar product (4.38) with the measure (4.30) given by

$$\mu_{N=2}(x) = \frac{1}{2\pi} \left| \frac{\Gamma^4(s+ix)}{\Gamma(2ix)\Gamma^3(2s)} \right|^2 . \tag{5.3}$$

This property is in a perfect agreement with the orthogonality condition for the Wilson polynomials [21].

The eigenvalue of the $N = 2$ hamiltonian corresponding to (5.1) can be calculated either by replacing the operator J_{12} in the expression for $\mathcal{H}_2$ by its eigenvalue $(J_{12} - h - 2s)\Psi_{p,h} = 0$, or by applying (3.38). In this way, one obtains

$$E_h = \pm i \frac{d}{d\epsilon} \ln Q_h(\pm is + \epsilon) \bigg|_{\epsilon=0} = 2\left[\psi(h+2s) - \psi(2s)\right] . \tag{5.4}$$

The ground state corresponds to $h = 0$.

Let us examine the unitary transformation to the SoV representation (4.1) for $N = 2$. It is defined by the transition kernel, eqs. (4.20) and (4.21), which looks at $N = 2$ like

$$U_{p,x}(z_1, z_2) = p^{2s-1/2} \int \mathcal{D}w_2 \; \mathrm{e}^{ipw_2}\, \Lambda_x(z_1, z_2|\bar{w}_2) , \tag{5.5}$$

where $\Lambda_x(z_1, z_2|\bar{w}_2)$ are given by (4.11)

$$\Lambda_x(z_1, z_2|\bar{w}_2) = \mathrm{e}^{3i\pi s} \int \mathcal{D}y_2\, (y_2 - \bar{w}_2)^{-\beta_x} (z_2 - \bar{w}_2)^{-\alpha_x} (z_1 - \bar{y}_2)^{-\alpha_x} (z_2 - \bar{y}_2)^{-\beta_x} , \tag{5.6}$$

with $\beta_x = s+ix$ and $\alpha_x = s-ix$. It is convenient to transform $U_{p,x}(z_1, z_2)$ to the momentum representation.

For arbitrary function $\Psi(z_1, z_2) \in \mathcal{V}_2$ this transformation is defined as

$$\Psi(z_1, z_2) = \frac{1}{\Gamma(2s)} \int_0^\infty dp_1\, dp_2\; \mathrm{e}^{i(p_1 z_1 + p_2 z_2)} (p_1\, p_2)^{s-1/2}\, \widetilde{\Psi}(p_1, p_2) , \tag{5.7}$$

where the additional factor $(p_1\, p_2)^{s-1/2}$ was introduced to simplify the expression for the scalar product (2.5) in the momentum representation (see eq. (A.3))

$$\langle \Psi_1 | \Psi_2 \rangle = \int_0^\infty dp_1 dp_2 \left(\widetilde{\Psi}_1(p_1, p_2) \right)^* \widetilde{\Psi}_2(p_1, p_2) . \tag{5.8}$$

In particular, the $N = 2$ eigenstates (5.1) are given in the momentum representation by the Jacobi polynomials[9]

$$\widetilde{\Psi}_{p,h}(p_1, p_2) = a_h\, \delta(p - p_1 - p_2)(p_1 p_2)^{s-1/2} (p_1 + p_2)^h\, \mathrm{P}_h^{(2s-1,2s-1)} \left(\frac{p_1 - p_2}{p_1 + p_2} \right) , \tag{5.9}$$

[9]This expression is well-known in QCD as defining the conformal operators built from two fields with the conformal spin s.

where $a_h = i^{-h-4s} h! \Gamma(2s)/\Gamma^2(h+2s)$ and delta-function ensures the momentum conservation. Applying (A.1) and performing integration in (5.5), one finds that in the momentum representation the $N=2$ transition kernel is given by

$$\widetilde{U}_{p,x}(p_1,p_2) = \delta(p-p_1-p_2)\,\frac{\Gamma^2(2s)\,\mathrm{e}^{i\pi s}}{|\Gamma(s+ix)|^2}\left(\frac{p}{p_1p_2}\right)^{1/2}\left(\frac{p_2}{p_1}\right)^{s}{}_2F_1\left(\left.\begin{matrix} s-ix, s+ix \\ 2s \end{matrix}\,\right|-\frac{p_2}{p_1}\right). \tag{5.10}$$

It defines the SoV transformation $\widetilde{\Psi}_p(p_1,p_2) \mapsto \Phi(x)$

$$\Phi(x)\delta(p-p') = \int_0^\infty dp_1\,dp_2\,\left(\widetilde{U}_{p',x}(p_1,p_2)\right)^*\,\widetilde{\Psi}_p(p_1,p_2)\,, \tag{5.11}$$

with $p, x > 0$. Substituting (5.10) into this relation and introducing notations for $\widetilde{\Psi}_p(p_1,p_2) = \delta(p-p_1-p_2)p^{-1/2}\xi^{s-1/2}(1+\xi)f(\xi)$ with $\xi = p_2/p_1$, one finds that (5.11) is reduced to

$$\Phi(x) = \mathrm{e}^{i\pi s}\,\frac{\Gamma^2(2s)}{|\Gamma(s+ix)|^2}\int_0^\infty d\xi\,\xi^{2s-1}\,{}_2F_1\left(\left.\begin{matrix} s-ix, s+ix \\ 2s \end{matrix}\,\right|-\xi\right)f(\xi)\,. \tag{5.12}$$

This relation defines the map $f(\xi) \mapsto \Phi(x)$, which is known as the Fourier-Jacobi or the index hypergeometric transform [19]. Then, the unitarity of the SoV transformation at $N=2$ follows from the similar property of the transformation (5.12).

Let us replace $\widetilde{\Psi}_p(p_1,p_2)$ in (5.11) by the $N=2$ eigenstate (5.9). According to (4.37) and (4.3), the corresponding eigenfunction in the separated variables is given by the Wilson polynomial $\Phi(x) = Q_h(x) \sim W_h(x^2, s, s, s, s)$. Then, (5.12) leads to a known representation for the Wilson polynomials as the index hypergeometric transform of the Jacobi polynomials [19].

6. Concluding remarks

In this paper we have constructed the Baxter $\mathbb{Q}$-operator and the representation of the Separated Variables for the open homogeneous $\mathrm{SL}(2,\mathbb{R})$ spin magnet. Our analysis relied on the diagrammatical approach developed in refs. [13] in application to the closed spin chain. In this approach, one represents the kernels of the relevant integral operators ($\mathbb{Q}$-operator, the transition kernel to the SoV representation) as Feynman diagrams and establishes their various properties with a help of a few simple diagrammatical identities.

We found that the Feynman diagrams for the $\mathbb{Q}$-operator and the transition kernel to the SoV representation have a remarkably simple form (see figures 1 and 7). In the latter case, the diagram reveals a universal pyramid-like structure which has been already observed for various quantum integrable models like periodic Toda chain [9], closed $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ spin chains [12, 13] and Calogero-Sutherland model [14]. This structure is a manifestation of a general factorization property (4.21) of the transition kernel to the SoV representation. Namely, the kernel is factorized into the product of Λ-operators each depending on a single separated variable. The only difference between the models mentioned above resides in the explicit form of the Λ-operator. The latter can be obtained as a certain limit of the $\mathbb{Q}$-operator leading to the expression for the transition kernel to

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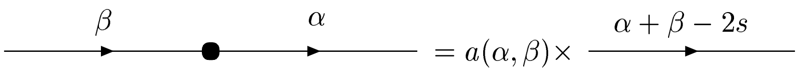

Figure 9: Chain relation.

the SoV representation as the product of the $\mathbb{Q}$-operators projected onto a special reference state. Another advantage of the diagrammatical approach is that it offers a simple regular way for calculating the integration measure in the SoV representation.

We found that there exists an intrinsic relation between the open spin chains and Wilson polynomials [21]. The latter occupy the top level in the Askey scheme of hypergeometric orthogonal polynomials [22]. These polynomials define the eigenvalues of the Baxter operator for open spin chain with $N = 2$ sites [4].

It is straightforward to extend our analysis to the case of inhomogeneous open $\mathrm{SL}(2,\mathbb{R})$ spin chains. One can show that for such models the transition kernel to the SoV representation is given by the same pyramid-like diagram shown in figure 7 with the only difference that both the indices attached to various lines and the integration measure corresponding to internal vertices should be modified appropriately. Another interesting possibility could be to consider an open spin chain with the $\mathrm{SL}(2,\mathbb{C})$ symmetry. Such models naturally appear in high-energy QCD as describing Regge singularities of scattering amplitudes with meson quantum numbers [23]. In that case, the quantum space of the model does not possess the highest weight (pseudovacuum state) and, therefore, the Algebraic Bethe Ansatz is not applicable.

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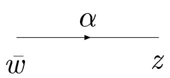
A. Feynman diagram technique

Here we collect some useful formulae for the $\mathrm{SL}(2,\mathbb{R})$ integrals. Their derivation can be found in ref. [13].

- Propagator:



$$= \frac{1}{(z-\bar{w})^{\alpha}} = \frac{\mathrm{e}^{-i\pi\alpha/2}}{\Gamma(\alpha)} \int_0^\infty dp\, \mathrm{e}^{ip\,(z-\bar{w})}\, p^{\alpha-1}\,. \tag{A.1}$$

- "Chain relation" (see figure 9):

$$\int \mathcal{D}w\, (z-\bar{w})^{-\alpha}(w-\bar{v})^{-\beta} = a(\alpha,\beta)\,(z-\bar{v})^{-\alpha-\beta+2s}\,, \qquad (\alpha+\beta\neq 2s) \tag{A.2}$$

- Delta-function relation:

$$\int \mathcal{D}w\, \mathrm{e}^{ipw-ip'\bar{w}} = \delta(p-p')\, p^{1-2s}\cdot\Gamma(2s)\,. \tag{A.3}$$

- Identity operator:

$$[\mathbb{K}\cdot\Psi](z) = \int \mathcal{D}w\, \frac{\mathrm{e}^{i\pi s}}{(z-\bar{w})^{2s}}\Psi(w) = \Psi(z)\,. \tag{A.4}$$

- Contour-integral representation:

$$\int \mathcal{D}w\, \frac{\mathrm{e}^{i\pi s}\,\Psi(w)}{(z_1-\bar{w})^{\alpha_x}(z_2-\bar{w})^{\beta_x}} = \frac{\Gamma(2s)}{\Gamma(\alpha_x)\Gamma(\beta_x)}\int_0^1 d\tau\tau^{\alpha_x-1}(1-\tau)^{\beta_x-1}\Psi(\tau z_1+(1-\tau)z_2)\,. \tag{A.5}$$

- Fourier integral:

$$\int \mathcal{D}w\, \frac{\mathrm{e}^{ipw}}{(z-\bar{w})^{\alpha}} = \theta(p)\, p^{\alpha-2s}\,\mathrm{e}^{ipz}\cdot\mathrm{e}^{-i\pi\alpha/2}\,\frac{\Gamma(2s)}{\Gamma(\alpha)}\,. \tag{A.6}$$

- Permutation identity (see figures 4 and 5):

$$(z_2-\bar{v}_2)^{i(x-y)}I(z,\bar{v}\,;x,y) = (z_1-\bar{v}_1)^{i(x-y)}I(z,\bar{v}\,;y,x)\,, \tag{A.7}$$

where $z=(z_1,z_2)$, $\bar{v}=(\bar{v}_1,\bar{v}_2)$ and

$$I(z,\bar{v}\,;x,y) = \int \mathcal{D}w \frac{1}{(w-\bar{v}_1)^{\alpha_x}(w-\bar{v}_2)^{\beta_x}(z_1-\bar{w})^{\beta_y}(z_2-\bar{w})^{\alpha_y}}\,. \tag{A.8}$$

In these relations, $\alpha_x = s-ix$ and $\beta_x = s+ix$ for arbitrary x, the a-function is defined in (4.19) and the integration measure $\mathcal{D}w$ is given by (2.6), $\bar{w}=w^*$ and $p>0$.

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